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UNIVERSITY OF CALIFORNIA

Los Angeles

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Advances in Filtering Techniques for Stochastic Systems with Uncertain Parameters . Part I.

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering

9 Doctoral thesis

by

10 Calvin Chris Schneider, Jr.

12 294p.

11 1978

PART I

9 FINAL REPORT

15 F04701-77-C-0090

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Advances in Filtering Techniques for Stochastic Systems with Uncertain Parameters

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering

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Calvin Chris Schneider, Jr.

1978

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University of California, Los Angeles

1978

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To Those Who Came With Me
For Pride

To Those Who Come After Me
For Inspiration

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LIST OF SYMBOLS

$\hat{\underline{b}}_1, \hat{\underline{b}}_2, \hat{\underline{b}}_3$	orthogonal unit vectors fixed in rotating body
B	continuous input matrix relating the input to time-derivative of the state
$d(\cdot)$	differential element $(\cdot)$
dm	element of body mass
e_i	error between the true state and the estimate state at time instant i (i.e., $x_i - \hat{x}_i$)
$E[\cdot]$	expectation operator of $[\cdot]$
$f(\cdot)$	function of $(\cdot)$
F	continuous state transition matrix relating the present state to its time-derivative
$F(\alpha,\alpha)$	Fisher information matrix used in adaptive parameter estimate
$F_i(\alpha,\alpha)$	element of Fisher information matrix accumulated at time instant i and used in adaptive parameter estimate
$g(\cdot)$	function of $(\cdot)$
G	continuous noise input matrix relating the noise to the time-derivative of the state
G_i	gain term of modified Kalman filter
$h(\cdot)$	function of $(\cdot)$

$H_i, H_i(\alpha)$	measurement matrix at a time instant i relating the state vector to the measurement vector and which may be a function of parameters α
H_i'	reformatted measurement matrix used in extended Kalman filter
i	time index
I	identity matrix
I_1, I_2, I_3	moments of inertia of principal body axes
j	time index
J_i	term of modified Kalman filter gain
$J(\alpha)$	variational terms of $-\ln p(\alpha \| Z_n)$
$_\alpha J_i(z_i, \alpha)$	derivative of one element of $J(\alpha)$ at time instant i with respect to parameters α
$_\alpha J(Z_N, \alpha)$	derivative of $J(\alpha)$ with respect to parameters α
$_\alpha \overline{J}_i(z_i, \alpha \| \alpha_t)$	expected value of $_\alpha J(z_i, \alpha)$ given α_t
$_\alpha J_i(Z_n, \alpha \| \alpha_t)$	expected value of $_\alpha J(Z_N, \alpha)$ given α_t
k	index of parameter α
K_i	Kalman gain at time instant i
K_i'	extended Kalman filter gain at time instant i
ℓ	index of parameter α
L	Liapunov testing function

$L[\cdot]$	log-likelihood function of $[\cdot]$
$L(\alpha)$	term utilized in Fisher information matrix $F(\alpha,\alpha)$
m	dimension of input vector u_i
M	angular momentum of spinning body
n	dimension of state vector x_i
N	number of measurements utilized to estimate the value of the parameters α
p	number of constant parameters in system equations which are not known precisely
$p(\cdot)$	probability density function of $(\cdot)$
$p(\cdot \mid \cdot\cdot)$	conditional probability density function of $(\cdot)$ given $(\cdot\cdot)$
P_i	covariance of state estimate error at time instant i after processing measurement z_i
$P_{i/i-1}$	covariance of state estimate error at time instant i after processing measurement z_{i-1}
P_i'	covariance of state estimate error of extended Kalman filter at time instant i after processing measurement z_i
$P_{i/i-1}'$	covariance of state estimate error of extended Kalman filter at time instant i after processing measurement z_{i-1}
q	dimension of noise input vector
Q_i	variance of noise input w_i at time instant i

r	dimension of measurement vector z_i
r_1, r_2, r_3	generic scalar components to the mass element dm along the $\hat{\underline{b}}_1$, $\hat{\underline{b}}_2$ or $\hat{\underline{b}}_3$ axes
$_{\alpha_k}r_i$	variational term of modified Kalman filter gain
R_i	variance of noise input v_i at time instant i
s_i	element of the score accumulated at time instant i and used in adaptive parameter estimate
$_{\alpha_k}s_i$	variational term of modified Kalman filter gain
S_i	sum of the product of $_{\alpha_k}H_i$ and $_{\alpha_k}s_i$ for all parameters from k equals 1 to p used in modified Kalman filter
$_{\alpha_k}S_i'$	symmetric matrix containing a derivative with respect to parameter α_k and used to evaluate Fisher information matrix
$_{\alpha_\ell}S_i''$	symmetric matrix containing a derivative with respect to parameter α_ℓ and used to evaluate Fisher information matrix
t_i	time at instant i
$_{\alpha_k}t_i$	variational term of state error co-variance of modified Kalman filter
T	kinetic energy of body
T_i	sum of $_{\alpha_k}t_i$ for all parameters from k equals 1 to p
u_i	input vector at time instant i

u_i'	input vector of reformatted equations of extended Kalman filter
v_i	additive noise vector to measurement equation at time instant i
V_i	variance of the innovation at time instant i
w_i	noise input vector of state equation at time instant i
$W(t,t_i)$	contribution of system noise input to state estimate error covariance from time t_i to t
x_i	state vector at time instant i
$\overline{x}_i$	filter prediction of x_i given measurements z_{i-1}
$\hat{x}_i$	filter estimate of x_i given measurements z_i
x_i^*	minimum error variance estimate of x_i
x_i'	state vector of extended Kalman filter at time instant i obtained by augmenting the state vector x_i with the parameters α
$\overline{x}_i'$	extended Kalman filter prediction of x_i'
$\hat{x}_i'$	extended Kalman filter estimate of x_i' given measurement z_i'
z_i	measurement vector at time instant i
z_i'	measurement vector at time instant i of reformatted equations for extended Kalman filter

LIST OF SYMBOLS (cont)

Z_i	vector of realized values of i measurements $z_1 \ldots z_i$
α	parameters existing in matrices of system and measurement equations
α^*	current estimate of parameters
α_o	nominal value of parameters
α_t	true value of parameters
β	vector process of independent Brownian motions
δ_{ij}	Kronecker delta function
$\delta\alpha_k$	variation of parameter α_k
ϵ_i	error between the true state and the predicted state at time instant i (i.e., $x_i - \overline{x}_i$)
θ	nutation angle of a spinning body defined as the angle between the spin axis and the angular momentum vector
λ	moment-of-inertia factor
Λ	vector of independent Gaussian variables with zero mean and unity variance
μ_i	bias term of modified Kalman filter
μ_i	innovation at time instant i defined as the difference between the actual measurement and the predicted value of the measurement

LIST OF SYMBOLS (cont)

σ	covariance of parameter error
$\phi_{i/i-1}, \phi(t_i t_{i-1}), \phi_{i/i-1}(\alpha)$	state transition matrix between time instants i and i-1 which is a function of parameters α
$\phi'_{i/i-1}$	reformatted state transition matrix used in extended Kalman filter
${}_{\alpha_k}\phi_{i/i-1}$	derivative of state transition matrix $\phi_{i/i-1}$ with respect to parameter α_k
$\psi_{i/i-1}, \psi(t_i t_{i-1}), \psi_{i/i-1}(\alpha)$	input matrix between time instants i and i-1 which is a function of parameters α
${}_{\alpha_k}\psi_{i/i-1}$	derivative of input matrix $\psi_{i/i-1}$ with respect to parameter α_k
$\Gamma_{i/i-1}, \Gamma_{i/i-1}(\alpha)$	noise input matrix between time instants i and i - 1 which is a function of parameters α
$\Gamma'_{i/i-1}$	reformatted noise input matrix used in extended Kalman filter
${}_{\alpha_k}\Gamma_{i/i-1}$	derivative of noise input matrix $\Gamma_{i/i-1}$ with respect to parameter α_k
Ω	frequency of rotation of body spin axis about angular momentum vector with respect to the body-fixed axes
$(\dot{\ })$	first time derivative of ( )
$(\ddot{\ })$	second time derivative of ( )

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ACKNOWLEDGMENTS

I wish to thank the members of the Doctoral Committee: Professor Cornelius T. Leondes, Chairman; Professor Kurt Forster; Professor Hugh H. Kieffer; Professor Margaret C. Kivelson; and Professor D. Lewis Mingori for their time, criticisms and suggestions which helped make this a better document. Professor Leondes was a constant source of encouragement throughout the research effort, and I am grateful for his invaluable guidance.

I am also grateful to my associates at The Aerospace Corporation, especially Dr. Joseph L. LeMay for his stimulating discussions and review of the research material and Dr. Stephen S. Bayliss, Dr. Ira A. Gura, and Mr. Thomas J. Lang for their assistance in overcoming some of the computer program difficulties of the research.

I also wish to acknowledge the aid and support received from the management of The Aerospace Corporation through the Advanced Study Grant Program and the pleasant association with the Program Administrator, Ms. Eleanor W. Anderson. Without this support, the research and completion of this dissertation would have been impossible. In a larger sense, I am also grateful to the whole aerospace

industry which promotes, encourages, and supports the advanced education of their employees. Such an atmosphere enabled me to acquire the background necessary to initiate this work.

Special thanks also goes to Ms. Peggy H. Ritter for the care and precision she exercised in typing and preparing this manuscript and to Ms. Virginia Burgess for her conscientious and careful editing of the manuscript.

Finally, I wish to thank my wife, Tonya, for her support and patience during this project and my children, Collette and Chris III, who each made sacrifices so that Daddy could do his school work.

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PUBLICATIONS

Schneider, C. C., and Likens, P. W. March 1973	Nutation dampers vs. precession dampers for asymmetric spinning spacecraft. Journal of Spacecraft and Rockets 10 (3):218-222.

ABSTRACT OF THE DISSERTATION

Advances in Filtering Techniques

for Stochastic Systems with Uncertain Parameters

by

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Professor Cornelius T. Leondes, Chairman

Optimum estimation of the states of linear dynamic processes using noise corrupted measurements, commonly referred to as Kalman filtering, requires an exact knowledge of the equations which govern the complete stochastic system. In practical applications, however, these equations depend on parameters which can not be precisely defined. When the parameter uncertainties are sufficiently large, standard filtering techniques can produce inaccurate and inadequate estimates. In this dissertation, two alternative techniques of state estimation for systems with uncertain parameters are considered.

The first technique assumes a modification to the Kalman filter estimate which incorporates the parameter uncertainty without estimating the parameter. The second technique defines an adaptive estimate of the uncertain parameters which is then utilized to improve the state estimates. Both techniques are applied to a numerical example. Computational results are generated and compared to estimates produced by a Kalman filter and an extended Kalman filter. The effects of the measurement noise level, a gap in the measurement data, and an erroneous data spike on the filter estimates are also considered.

CHAPTER 1.0

INTRODUCTION

1.1 LINEAR FILTERING

Linear filtering is concerned with finding an optimal estimate of some quantity (an unknown parameter, a random variable, or a random signal) when an adequate system model is a linear function of that quantity. The linear equation may be corrupted by additive noise and/or driven by known inputs.

One of the first studies dealing with linear estimation was performed in the early 1800's by Gauss (1963) who developed the method of least-squares to estimate unknown parameters. Legendre is also considered as an original inventor of the same theory based on his work in 1806 titled, "Nouvelles Methodes pour la Determination des Orbites des Cometes," (see Sorenson 1970). The motivation in the development of estimation theory during this period was based on the desire to calculate orbital parameters of the motions of heavenly bodies. In these astronomical studies, the orbiting bodies were observed using telescopes and values of the unknown parameters were inferred from the measurement data.

The next significant development in filtering theory was achieved in the 1940's by Kolmogorov (1941) and Wiener (1949) for the class of problems in which the signal and measurements were scalar, random processes. Kolmogorov studied only discrete-time problems of stationary random processes and solved them by using a simple representation of such processes that was suggested in a doctoral dissertation by Wold (1938). This representation, which is obtained by a recursive orthonormalization procedure, is known as the Wold decomposition. Wiener, on the other hand, studied mainly continuous-time problems and took an almost non-probabilistic approach. The main result of his work was an integral equation called the Wiener-Hopf equation. The solution of this equation is a weighting function which, when convolved with the corrupted linear measurement, produces an unbiased minimum variance estimate of the random signal. The Wiener-Hopf equation, however, can only be solved explicitly for certain stationary random processes. For this reason, general filter design using the Wiener-Hopf integral equation has only limited practical applications. Many generalizations of this work were pursued in the 1940's and 1950's by Blackman, Bode, and Shannon (1948), Bode and Shannon (1950), Doob (1953), and Lattin and

Battin (1958) but very few practical results were achieved, notable exceptions being the work of Booton (1952) and Zadeh and Ragazzini (1950, 1952).

In order to circumvent the difficulty of solving the integral equation for the general filtering problem, the idea of generating least-squares estimates in a recursive manner was introduced in the 1950's. This interest was stimulated by the development and increased use of digital computers. With the digital computer, the estimates were generated dynamically by a data processing algorithm which represented either a differential or difference equation. The first works of recursive least-squares estimations were produced by Carlton and Follin (1956) and Swerling (1959). Swerling's report presented a recursive filtering procedure similar to that described shortly thereafter by Kalman (1960) and is generally considered to be the starting point of modern estimation theory commonly referred to as "Kalman filtering."

The paper by Kalman and a subsequent paper by Kalman and Bucy (1961) generalized the results of linear estimation theory. The main contribution of these papers was the transformation of the Wiener-Hopf integral equation into an equivalent nonlinear differential equation whose

solution yields the covariance matrix of the minimum filtering error and provides all the information necessary for the design of the optimal filter. Thus, by requiring a numerical rather than an analytical solution to the Wiener-Hopf equation, a simple recursive filter was developed which could be easily synthesized on a digital computer.

The optimality of the Kalman filter, however, is dependent upon complete knowledge of the parameters which define the system. In practical applications, these quantities may not be precisely defined. When the parameter uncertainties are sufficiently large, the standard Kalman filter can produce inaccurate and inadequate estimates.

1.2 SYSTEM MODEL AND PROBLEM STATEMENT

Consider a dynamic system represented by a linear difference equation and driven by a combination of known inputs and white noise. The system may be a discrete-time process or a continuous-time process modeled in discrete time by means of a state transition matrix. The stochastic, vector difference equation for such a system is given by

$$x_{i+1} = \phi_{i+1/i}(\alpha)\, x_i + \psi_{i+1/i}(\alpha)\, u_i + \Gamma_{i+1/i}(\alpha) w_i \tag{1.1}$$

where

x_i - n-dimensional state vector at time instant i

u_i - m-dimensional input vector at time instant i

w_i - q-dimensional noise vector at time instant i

$\phi_{i+1/i}(\alpha)$ - n x n transition matrix between time instants i and i+1

$\psi_{i+1/i}(\alpha)$ - n x m input matrix between time instants i and i+1

$\Gamma_{i+1/i}(\alpha)$ - n x q noise input matrix between time instants i and i+1

and where each of the matrices $\phi_{i+1/i}(\alpha)$, $\psi_{i+1/i}(\alpha)$, and $\Gamma_{i+1/i}(\alpha)$ may depend on p constant parameters designated α. The information available about the state is obtained from measurements related to the state according to

$$z_i = H_i(\alpha)x_i + v_i \tag{1.2}$$

where

z_i - r-dimensional measurement vector at time instant i

v_i - r-dimensional noise vector at time instant i

$H_i(\alpha)$ - r x n measurement matrix at time instant i

and where the measurement matrix $H_i(\alpha)$ may also depend on the parameters α.

Although the same parameters may not exist jointly in the measurement, transition, input, and noise input matrices, $H_i(\alpha)$, $\phi_{i/i-1}(\alpha)$, $\psi_{i/i-1}(\alpha)$, and $\Gamma_{i/i-1}(\alpha)$ respectively, this representation is assumed here to generalize the results to follow. The case involving different parameters in each of the matrices is then a subset of this general representation. Note that if the actual system is a continuous-time system, as described by the equation

$$\dot{x} = Fx + Bu + Gw \tag{1.3}$$

where

F - continuous state transition matrix

B - continuous input matrix

G - continuous noise input matrix

then a single uncertain parameter in F will generally affect both $\phi_{i/i-1}(\alpha)$ and $\psi_{i/i-1}(\alpha)$ in the equivalent discrete representation (see Appendix A).

The noise vectors, w_i and v_i, are assumed to be uncorrelated Gaussian white noise sequences with statistics

$$E[w_i] = 0 \tag{1.4}$$

$$E[v_i] = 0 \tag{1.5}$$

$$E\left[w_i v_j^T\right] = 0 \text{ for all } i,j \tag{1.6}$$

$$E\left[w_i w_j^T\right] = Q_i \delta_{ij} \tag{1.7}$$

$$E\left[v_i v_j^T\right] = R_i \delta_{ij} \tag{1.8}$$

where $E[\cdot]$ denotes the expectation operator, Q_i is a positive definite q x q matrix, R_i is a positive definite r x r matrix, and δ_{ij} is the Kronecker delta function.

The initial state x_o is defined as a Gaussian random vector defined by a mean $\hat{x}_o$ and covariance P_o as follows:

$$E[x_o] = \hat{x}_o \tag{1.9}$$

$$E\left[(x_o-\hat{x}_o)(x_o-\hat{x}_o)^T\right] = P_o \tag{1.10}$$

where P_o is a positive definite n x n matrix.

The true values for the parameters in the system model are assumed to be unknown, the expected time-invariant value being α_o. The variation $\delta\alpha$(i.e., $\alpha-\alpha_o$) is assumed to be a normally distributed vector of zero mean and variance defined by

$$E\left[(\delta\alpha)(\delta\alpha)^T\right] = \sigma = \text{diag}\left[\sigma_1, \sigma_2, \ldots\sigma_p\right] \quad (1.11)$$

where σ is a matrix with elements along the diagonal designated σ_1 through σ_p.

It should be noted that parameter uncertainties in the measurement matrix are generally small or nonexistent in most problems compared with uncertainties in the transition and input matrices. However, they are assumed here to provide a more general development.

The problem is to obtain consistent and improved estimates of the states x_i considering the uncertainty of the parameters α, the objective being the development of advanced techniques of state estimation which avoid the difficulties of previous approaches.

1.3 PREVIOUS RESEARCH

Originally, the problem of state estimation for stochastic systems with uncertain parameters was posed as a

nonlinear estimation problem by augmenting the state vector with the unknown parameters in the state equations. The extended Kalman-Bucy filter was then applied to the resulting nonlinear filtering problem by Kopp and Orford (1963), Farison, Graham, and Shelton (1967), and Jazwinski (1970). Although this technique is feasible, the computational burden is increased, and the filter is subject to divergence. The divergence is due primarily to the linearization about the current estimate of the extended Kalman filter. If the a priori estimates are poor or if the filtered estimates lie outside the linear region, the filter will often diverge.

Another approach by Neal (1967) has been to artificially increase selected elements of the plant noise covariance matrix to reflect the increased uncertainty in the state dynamics. However, this trick is a process of judiciously tuning the Kalman filter and becomes more of an art than a science. Recently, a trajectory sensitivity approach was taken by Chang and Belanger (1976) to the design of the Kalman filter gain for a continuous-time system. A two-point boundary value problem was formulated where the performance index was a quadratic function of the final error. Through expedient approximations, a solution

was obtained and the final estimate was more accurate than that of the nominal Kalman filter; however, intermediate estimates did not show as much improvement.

In adaptive filtering, the problem has been formulated using decision theory by Magill (1965) and Lainiotis (1971) and recently using maximum likelihood by Maybeck (1968), Mehra (1970), and Gupta and Mehra (1974). The decision theory approach assumes a bank of parallel Kalman filters with each filter matched to a possible parameter vector value. The state estimates generated by the Kalman filters are then combined using a weighted sum in which the Bayesian a posteriori hypothesis probabilities are weighting factors. If one of the selected parameter vectors coincides with the true parameter, this method gives the minimum variance estimates of both the state and parameters. The computational burden of this method, however, can be quite significant. The maximum likelihood estimation is based on the innovation sequence of the system. However, since the parameters and the parameter probability distribution are both assumed to be unknown, the computation aspects of the solution can be extremely difficult.

1.4 PROPOSED APPROACH

The research presented herein consists of developing

advanced techniques of state estimation for stochastic systems with uncertain parameters. Both nonadaptive and adaptive techniques are considered. The nonadaptive approach assumes a Kalman-type filter defined as

$$\hat{x}_i = \bar{x}_i + G_i \left| z_i - H_i(\alpha_o)\bar{x}_i \right| + \mu_i \tag{1.12}$$

where

$$\bar{x}_i = \phi_{i/i-1}(\alpha_o)\hat{x}_{i-1} + \psi_{i/i-1}(\alpha_o)u_{i-1} \tag{1.13}$$

and

$\bar{x}_i$ - predicted value for x_i

G_i - modified Kalman gain matrix

μ_i - estimator bias

The transition matrices of the state equation are expanded to first order in the α parameters. The state equation becomes

$$x_i \approx x_i(\alpha_o) + \sum_{k=1}^{p} \frac{dx_i}{d\alpha_k} \delta\alpha_k \tag{1.14}$$

where

$$x_i(\alpha_o) = \phi_{i/i-1}(\alpha_o)x_{i-1} + \psi_{i/i-1}(\alpha_o)u_{i-1} + \Gamma_{i/i-1}(\alpha_o)w_{i-1} \tag{1.15}$$

$$\frac{dx_i}{d\alpha_k} = \frac{d\phi_{i/i-1}}{d\alpha_k} x_{i-1} + \frac{d\psi_{i/i-1}}{d\alpha_k} u_{i-1} + \frac{d\Gamma_{i/i-1}}{d\alpha_k} w_{i-1} \tag{1.16}$$

The bias of the estimator is then determined by requiring that the expected value of the error e_i (i.e., $x_i-\hat{x}_i$) be zero or

$$\mu_i = f(E[e_i] = 0) \tag{1.17}$$

The filter gain is defined by requiring that the expected value of a quadratic function of the estimation error is minimized. Equivalently, this is given by

$$G_i = f\left(\frac{\partial E\left[e_i^T e_i\right]}{\partial G_i} = 0\right) \tag{1.18}$$

In the adaptive technique, a Kalman filter is coupled with a Bayesian estimator of the uncertain

parameters. The Bayesian estimator is a maximum a posteriori estimator obtained by maximizing the conditional density function of the parameters given by

$$\alpha = f\left(\frac{\delta p(\alpha|Z)}{\delta\alpha}\right) = 0 \qquad (1.19)$$

where $p(\alpha|Z)$ is the conditional density function of the parameters given the measurements Z. The state estimate is then determined using the estimate of the parameters. The state estimate is given by the standard Kalman filter equation

$$\hat{x}_i = \bar{x}_i + K_i(z_i - H_i\bar{x}_i) \qquad (1.20)$$

where K_i is the Kalman gain, and where all the terms that contain the parameters α are determined using the estimate α.

1.5 SUMMARY OF DISSERTATION

Chapter 2 develops the Kalman filter algorithm for linear, discrete processes. The problem of uncertain parameters in the system equation is then posed as a nonlinear problem by adjoining the uncertain parameters to the state

vector, and the extended Kalman filter is derived as a solution to this nonlinear problem.

In Chapters 3 and 4, research on alternative methods of state estimation for stochastic systems is presented. Chapter 3 provides the derivation of a modified Kalman filter, and complete estimation algorithms are developed. A discussion of the stability of the modified filter is also provided. Chapter 4 develops an adaptive estimator of the parameters, which is then utilized to improve the state estimates. In addition, the asymptotic nature of the parameter estimator is assessed.

The techniques presented in this dissertation are applied to a practical problem in Chapter 5. Computational results are generated for the filters proposed in Chapters 3 and 4. A comparison of these results with those produced by the Kalman filter and the extended Kalman filter in Chapter 2 is also presented. Finally, Chapter 6 presents the conclusions of the research and suggestions for future investigations.

CHAPTER 2

KALMAN FILTER AND EXTENDED KALMAN FILTER

This chapter develops the Kalman and extended Kalman filters for solving the problem of state estimation from samples of noise-corrupted data. The development is based on the assumption that all parameters of the system structure are known precisely. Section 2.1 derives the Kalman filter for systems represented by linear difference equations. The maximum likelihood concept is utilized to define the likelihood function from the available statistical information and this function generates the likelihood equation. The solution of the likelihood equation provides the desired Kalman filter expression. In Section 2.2, the problem of uncertain parameters in the system equations is considered. The linear problem is transformed to a nonlinear problem by augmenting the uncertain parameters to the state vector. The extended Kalman filter is then derived to solve this nonlinear problem.

2.1 OPTIMUM STATE ESTIMATION

This section develops the equations of the Kalman filter, the statistics of the filter estimation error, the statistics of the measurement information, and the

statistics of the measurement residual. The system model used for this development was presented in the previous chapter with the assumption that the true value of the parameters in the model are known.

2.1.1 Derivation of the State Estimate

Let Z_i denote the vector of realized values of the measurements $z_1 \ldots z_i$; thus, Z_i is a vector of dimension $(i \cdot m)$. In order to obtain a maximum likelihood estimate of the system state, a likelihood function must be defined relating the measurements (whose values are known) and the state variables (the unknowns to be estimated). An appropriate likelihood function is the conditional probability density of the state given the measurements $p(x_i|Z_i)$. (Note that all system parameters are implicitly assumed to be known.)

To specify the maximum likelihood estimate, the value x_i is sought that will yield the greatest possible value of the likelihood function. In this particular case, x_i is the value that will yield the peak of the conditional density function $p(x_i|Z_i)$, i.e., it will maximize the probability of the variables to be estimated, conditioned upon the events actually observed. Rather than directly maximize $p(x_i|Z_i)$, it is more convenient to find the

minimum of the negative natural logorithm of $p(x_i, Z_i)$. This is a valid step because for any function p, p and -ln(p) attain their maxima and minima, respectively, at the same point. Thus, the log-likelihood function is defined as

$$L(x_i, Z_i) = -\ln p(x_i | Z_i) \tag{2.1}$$

in order to evaluate the desired state estimate.

Using Bayes' rule, the conditional density is first written into a more convenient form:

$$p(x_i | Z_i) = \frac{p(x_i, Z_i)}{p(Z_i)}$$

$$= \frac{p(z_i | x_i, Z_{i-1})\ p(x_i, Z_{i-1})}{p(z_i | Z_{i-1})\ \ \ p(Z_{i-1})}$$

$$= \left[\frac{p(z_i | x_i, Z_{i-1})}{p(z_i | Z_{i-1})}\right] p(x_i | Z_{i-1}) \tag{2.2}$$

By means of this equation, the conditional density for x_i, given all measurements through z_i, is directly related to the density conditioned on all measurements up to, but not

including, z_i. It is used to relate the estimate of x_i just after the measurement z_i to the estimate just before z_i. The component terms of the above equation, however, must first be evaluated.

Let the mean of the density $p(x_i | Z_{i-1})$ be denoted as $\overline{x}_i$ and the corresponding conditional covariance as $P_{i/i-1}$. Then,

$$\overline{x}_i = E\left[x_i | Z_{i-1}\right] \tag{2.3}$$

$$P_{i/i-1} = E\left[(x_i-\overline{x}_i)(x_i-\overline{x}_i)^T | Z_{i-1}\right] \tag{2.4}$$

These expectations are ensemble averages over all possible initial conditions and both driving and measurement noise sequences, all conditioned on the values of the measurements. Because the system is linear, driven by a white Gaussian noise sequence and/or a deterministic input, and the probability density function of the initial state is Gaussian, the conditional probability density function of x_i before the measurement at instant i is normal. Thus, the density function is given by the equation

$$p(x_i|Z_{i-1}) = \frac{1}{(2\pi)^{n/2}\,|P_{i/i-1}|^{\frac{1}{2}}} \exp\left|\cdot\right|$$

$$\left|\cdot\right| = \left\{-\tfrac{1}{2}[x_i-\bar{x}_i]^T P^{-1}_{i/i-1}[x_i-\bar{x}_i]\right\} \qquad (2.5)$$

where $|P_{i/i-1}|$ is the determinant of $P_{i/i-1}$.

Consider now the term $p(z_i|x_i,Z_{i-1})$. The observation at time i is

$$z_i = H_i x_i + v_i \qquad (1.2)$$

(Note that the term $H_i(\alpha)$ is shortened to H_i since the parameters are assumed to be known.) The v_i sequence has been defined to be normally distributed and is independent of x_i. Therefore, conditioned on a particular value of x_i, z_i is a normal random variable with conditional mean

$$E\left[z_i|x_i,Z_{i-1}\right] = E\left[z_i|x_i\right]$$

$$= H_i x_i \qquad (2.6)$$

and conditional covariance

$$E\left[[z_i - H_i x_i][z_i - H_i x_i]^T \middle| x_i, Z_{i-1}\right] = E\left[v_i v_i^T\right] = R_i \quad (2.7)$$

so that the conditional probability density of z_i, given x_i and Z_{i-1}, is

$$p(z_i | x_i, Z_{i-1}) = \frac{1}{(2\pi)^{r/2} |R_i|^{\frac{1}{2}}} \exp\left|\cdot\right|$$

$$\left|\cdot\right| = \left|-\tfrac{1}{2}[z_i - H_i x_i]^T R_i^{-1} [z_i - H_i x_i]\right| \quad (2.8)$$

Finally, consider $p(z_i | Z_{i-1})$. The output of the dynamic equation is a Gauss-Markov sequence; in other words, x_i is a Gaussian random variable. Furthermore, linear operations on Gaussian random variables produce Gaussian random variables, so $H_i x_i$ is normally distributed. The sum of two independent normal random variables (i.e., $H_i x_i$ and v_i) yields a normal variable. Therefore, $p(z_i | Z_{i-1})$ is a normal density characterized by conditional mean and covariance:

$$E\left[z_i | Z_{i-1}\right] = H_i E\left[x_i | Z_{i-1}\right] + E\left[v_i | Z_{i-1}\right] = H_i \overline{x}_i \quad (2.9)$$

$$E\left[(z_i - H_i \overline{x}_i)(z_i - H_i \overline{x}_i)^T | Z_{i-1}\right] = H_i P_{i/i-1} H_i^T + R_i \quad (2.10)$$

and given by the equation

$$p(z_i|Z_{i-1}) = \frac{1}{(2\pi)^{r/2}\left|H_iP_{i/i-1}H_i^T + R_i\right|^{\frac{1}{2}}} \exp\left|\cdot\right|$$

$$\left|\cdot\right| = \left|-\tfrac{1}{2}(z_i-H_i\overline{x}_i)^T(H_iP_{i/i-1}H_i^T + R_i)^{-1} \cdot (z_i-H_i\overline{x}_i)\right. \tag{2.11}$$

Parenthetically, note that this term is not a function of x_i and therefore will have no effect on the derivation.

By incorporating the above equations into the expression for $p(x_i|Z_i)$ and taking the natural logarithm, the negative log-likelihood function is obtained as

$$\begin{aligned} L(x_i,Z_i) = {} & \tfrac{1}{2}\ln\left|(2\pi)^n\left|H_iP_{i/i-1}H_i^T + R_i\right|^{-1}\left|R_i\right|\,\left|P_{i/i-1}\right|\right| \\ & + \tfrac{1}{2}\left|(x_i-\overline{x}_i)^TP_{i/i-1}^{-1}(x_i-\overline{x}_i)\right| \\ & + \tfrac{1}{2}\left|(z_i-H_ix_i)^TR_i^{-1}(z_i-H_ix_i)\right| \\ & - \tfrac{1}{2}\left|(z_i-H_i\overline{x}_i)^T(H_iP_{i/i-1}H_i^T + R_i)^{-1}(z_i-H_i\overline{x}_i)\right| \end{aligned} \tag{2.12}$$

where the first and last terms do not involve x_i.

To obtain the maximum likelihood estimate of x(i), the equation

$$\left.\frac{\partial L(x_i, Z_i)}{\partial x_i}\right|_{x_i \rightarrow \hat{x}_i} = 0 \tag{2.13}$$

is solved, where $\hat{x}_i$ denotes the maximum likelihood estimate of x_i. Substituting $L(x_i, Z_i)$ into this equation yields

$$\left. P_{i/i-1}^{-1}(x_i - \bar{x}_i) - H_i^T R_i^{-1}(z_i - H_i x_i) \right|_{x_i \rightarrow \hat{x}_i} = 0 \tag{2.14}$$

The solution is then

$$\hat{x}_i = (P_{i/i-1}^{-1} + H_i^T R_i^{-1} H_i)^{-1}(P_{i/i-1}^{-1}\bar{x}_i + H_i^T R_i^{-1} z_i) \tag{2.15}$$

Instead of solving the likelihood Equation (2.13) directly by using Equation (2.12), algebraic minipulations (completing squares and applying the Householder inversion lemma) can first be performed upon the negative log-likelihood function to obtain the form

$$L[x(i),Z(i)] = \tfrac{1}{2}\ln\left|(2\pi)^n|P_i|\right|$$

$$+ \tfrac{1}{2}(x_i-\hat{x}_i)^T P_i^{-1}(x_i-\hat{x}_i) \qquad (2.16)$$

where

$$\hat{x}_i = (P^{-1}_{i/i-1} + H_i^T R_i^{-1} H_i)^{-1} \cdot (P^{-1}_{i/i-1}\bar{x}_i + H_i^T R_i^{-1} z_i) \qquad (2.17)$$

$$P_i^{-1} = P^{-1}_{i/i-1} + H_i^T R_i^{-1} H_i \qquad (2.18)$$

This demonstrates that $p(x_i|Z_i)$ is a normal density with mean x_i and covariance P_i:

$$p(x_i|Z_i) = \frac{1}{(2\pi)^{n/2}|P_i|^{1/2}} \exp\left|\cdot\right|$$

$$\left|\cdot\right| = \left|-\tfrac{1}{2}(x_i-\hat{x}_i)^T P_i^{-1}(x_i-\hat{x}_i)\right| \qquad (2.19)$$

$$\hat{x}_i = E[x_i|Z_i] \qquad (2.20)$$

$$P_i = E_i[(x_i-\hat{x}_i)(x_i-\hat{x}_i)^T] \qquad (2.21)$$

As expected, due to the symmetry of this density, the maximum likelihood estimate and the mean of the density coincide. The matrix inversion lemma can be used to put this estimate into a better computational form, requiring inversion of an m x m matrix instead of an n x n (m is typically smaller than n and, since measurements can be incorporated one at a time, the inversion can become a simple division). For positive definite $P_{i/i-1}$ and R_i, the lemma yields

$$(P_{i/i-1}^{-1} + H_i^T R_i^{-1} H_i)^{-1} = P_{i/i-1} - P_{i/i-1} H_i^T (H_i P_{i/i-1} H_i^T + R_i)^{-1} H_i P_{i/i-1} \tag{2.22}$$

so that the state estimate can be written as

$$\hat{x}_i = \bar{x}_i + K_i (z_i - H_i \bar{x}_i) \tag{1.20}$$

where

$$K_i = P_{i/i-1} H_i^T (H_i P_{i/i-1} H_i^T + R_i)^{-1} \tag{2.23}$$

which can be recognized as the form of the Kalman filter with K_i as the Kalman gain. The update of the covariance matrix from just before the i^{th} measurement to just after is given by Equation (2.18) rewritten as

$$P_i = (P_{i/i-1}^{-1} + H_i^T R_i^{-1} H_i)^{-1} \tag{2.24}$$

which, although it adds the measurement information in a simple way, requires an n x n matrix inversion. Equation (2.22) yields

$$P_i = P_{i/i-1} - P_{i/i-1} H_i^T (H_i P_{i/i-1} H_i^T + R_i)^{-1} H_i P_{i/i-1} \tag{2.25}$$

This expression can also be written

$$P_i = (I - K_i H_i) P_{i/i-1} \tag{2.26}$$

Although these forms involve m x m inversions, they do have some undesirable characteristics. Equation (2.25) is sometimes a small difference of large numbers (especially if the measurements are very accurate) and thus is subject to numerical precision problems. Equation (2.26) does not assure positive definiteness or symmetry, and thus can also

lead to numerical difficulties. Now, if the estimate in Equation (1.20) is rewritten as

$$\hat{x}_i = (I - K_i H_i)\bar{x}_i + K_i z_i \tag{2.27}$$

then it can readily be shown that an equivalent expression for P_i would be

$$P_i = (I - K_i H_i) P_{i/i-1} (I - K_i H_i)^T + K_i R_i K_i^T \tag{2.28}$$

This is the sum of two symmetric, positive definite matrices so numerical computations based upon this form will be better conditioned, better assuring the symmetry and positive definiteness of P_i.

The probability densities involved are normal, and the propagation of the entire density function can be specified by the time history of its mean and covariance. Thus, estimates propagated between measurements are

$$\bar{x}_{i+1} = E[x_{i+1} | Z_i] \tag{2.29}$$

or

$$\bar{x}_{i+1} = \phi_{i+1/i}\hat{x}_i + \psi_{i+1/i}u_i \tag{2.30}$$

and

$$P_{i+1/i} = E\left[(x_{i+1}-\bar{x}_{i+1})(x_{i+1}-\bar{x}_{i+1})^T \middle| Z_i\right]$$

$$= E\left[\left\{\phi_{i+1/i}(x_i-\hat{x}_i) + \Gamma_{i+1/i}w_i\right\}\cdot\right.$$

$$\left.\cdot\left\{\phi_{i+1/i}(x_i-\hat{x}_i) + \Gamma_{i+1/i}w_i\right\}^T \middle| Z_i\right] \tag{2.31}$$

or

$$P_{i+1/i} = \phi_{i+1/i}P_i\phi^T_{i+1/i} + \Gamma_{i+1/i}Q_i\Gamma^T_{i+1/i} \tag{2.32}$$

where $\phi_{i+1/i}(\alpha)$, $\psi_{i+1/i}(\alpha)$, and $\Gamma_{i+1/i}(\alpha)$ are written $\phi_{i+1/i}$, $\psi_{i+1/i}$, and $\Gamma_{i+1/i}$, respectively, since the parameters α are assumed to be known.

To summarize, at a measurement, the estimate is updated using:

$$\hat{x}_i = (I-K_iH_i)\bar{x}_i + K_iz_i \tag{2.27}$$

$$P_i = (I-K_iH_i)P_{i/i-1}(I-K_iH_i)^T + K_iR_iK_i^T \tag{2.28}$$

where

$$K_i = P_{i/i-1}H_i^T(H_iP_{i/i-1}H_i^T + R_i)^{-1} \tag{2.23}$$

and, to propagate the estimate up to the time of the next measurement, the relations are

$$\bar{x}_{i+1} = \phi_{i+1/i}\hat{x}_i + \psi_{i+1/i}u_i \tag{2.30}$$

$$P_{i+1/i} = \phi_{i+1/i}P_i\phi^T_{i+1/i} + \Gamma_{i+1/i}Q_i\Gamma^T_{i+1/i} \tag{2.32}$$

Note that K_i can also be expressed as

$$K_i = P_iH_i^TR_i^{-1} \tag{2.33}$$

(This can be expanded using the expression for P_i to show it is equal to the previous evaluation of K_i.) Although it is a simpler expression and has the same appearance as the continuous-time Kalman gain, it is not as convenient to use in recursions (see Ho 1962).

2.1.2 Statistics of the Estimation Error

This section defines the statistics of the error committed by the estimate immediately after the measurement

at time i. The error is given by the equation

$$e_i = x_i - \hat{x}_i \tag{2.34}$$

Since both x_i and $\hat{x}_i$ are Gaussian, e_i is also a Gaussian distribution, and thus its density function is completely specified by its mean and covariance, written as

$$E\left[e_i | Z_i\right] = \hat{x}_i - \hat{x}_i = 0 \tag{2.35}$$

$$E\left[e_i e_i^T | Z_i\right] = P_i \tag{2.36}$$

Thus the desired density can be written as

$$p(e_i | Z_i) = \frac{1}{(2\pi)^{n/2} |P_i|^{\frac{1}{2}}} \exp\left|-\tfrac{1}{2} e_i^T P_i^{-1} e_i\right| \tag{2.37}$$

Since P_i can be propagated independent of the particular sequence of measurements, using Equations (2.28) and (2.32), and since e_i is a dummy variable in Equation (2.37), it can be seen that $p(e_i | Z_i)$ is in fact equivalent to $p(e_i)$. In other words, e_i is statistically dependent upon the R_j for j equals 1 to i but is independent of the particular Z_i.

It should be recognized also that, since the expected value of e_i is zero, the state estimate $\hat{x}_i$ is unbiased.

2.1.3 Statistics of the Measurement Information

Consider the statistics of the new state information introduced at each measurement. From Equation (1.20), this information is portrayed by

$$\hat{x}_i - \overline{x}_i = K_i(z_i - H_i\overline{x}_i) \tag{2.38}$$

Upon substitution of z_i, this equation becomes

$$\begin{aligned}\hat{x}_i - \overline{x}_i &= K_i(H_i x_i - H_i\overline{x}_i + v_i)\\ &= K_i\Big| H_i\phi_{i/i-1}(x_{i-1}-\hat{x}_{i-1})\\ &\quad + H_i\Gamma_{i/i-1}w_{i-1} + v_i\Big|\end{aligned} \tag{2.39}$$

where the term in brackets is the sum of three independent terms, each independent of Z_{i-1}. Consequently, the value $(\hat{x}_i - \overline{x}_i)$ is independent of Z_{i-1}. Thus, the statistics of state information added by a measurement are expressed as:

$$E\left[(\hat{x}_i-\bar{x}_i)\; Z_{i-1}\right] = E\left[\hat{x}_i-\bar{x}_i\right]$$

$$= \underline{0} \qquad (2.40)$$

$$E\left[(\hat{x}_i-\bar{x}_i)(\hat{x}_i-\bar{x}_i)^T | Z_{i-1}\right] = E\left[(\hat{x}_i-\bar{x}_i)(\hat{x}_i-\bar{x}_i)^T\right]$$

$$= K_i(H_iP_{i/i-1}H_i^T + R_i)K_i^T$$

$$= P_{i/i-1}H_i^T(H_iP_{i/i-1}H_i^T + R_i)^{-1}H_iP_{i/i-1}$$

$$= K_iH_iP_{i/i-1} \qquad (2.41)$$

Recognize that the state information covariance is the quantity that reduces the state estimate error covariance at each measurement (see Equation 2.26)

2.1.4 Statistics of the Measurement Residual

The measurement residual at time is is the difference between the measurement z_i and the best prediction of z_i based on Z_{i-1}, $H_i\bar{x}_i$. This quantity, given by the equation

$$\nu_i = z_i - H_i\bar{x}_i \qquad (2.42)$$

may be regarded as the new information from the current observation z_i, given all the past observations Z_{i-1} and the past information deduced from them. As such, the term ν_i is commonly referred to as the innovation and the series of residuals is defined as the innovation sequence. This designation was first used by Wiener to describe such processes.

The innovation represents a linear operation on Gaussian random variables and, therefore, is itself a Gaussian random variable. The mean $\bar{\nu}_i$ and variance of the innovation V_i are given by

$$\bar{\nu}_i = E(z_i - H_i\bar{x}_i | Z_{i-1})$$

$$= E(H_i x_i | Z_{i-1}) + E(v_i | Z_{i-1}) - E(H_i\bar{x}_i | Z_{i-1})$$

$$= H_i\bar{x}_i - H_i\bar{x}_i$$

$$= 0 \tag{2.43}$$

$$V_i = E\left[(z_i - H_i\bar{x}_i)(z_i - H_i\bar{x}_i)^T | Z_{i-1}\right]$$

(cont)

$$= E\left[H_i(x_i-\bar{x}_i)(x_i-\bar{x}_i)^T H_i^T | Z_{i-1}\right] + E(v_i v_i^T | Z_{i-1})$$

$$= H_i P_{i/i-1} H_i^T + R_i \tag{2.44}$$

The correlation function of the innovation process is given by

$$E(\nu_i \nu_{i-j}^T) = E\left[(z_i - H_i\bar{x}_i)(z_{i-j} - H_{i-j}\bar{x}_{i-j})^T\right] \tag{2.45}$$

Utilizing the definition for the error in the predicted state

$$\epsilon_i = x_i - \bar{x}_i \tag{2.46}$$

this function becomes

$$E(\nu_i \nu_{i-j}^T) = H_i E\left[\epsilon_i \epsilon_{i-j}^T\right] H_{i-j}^T + H_i E\left[\epsilon_i v_{i-j}^T\right]$$

$$+ E\left[v_i \epsilon_{i-j}^T\right] H_{i-j}^T + E\left[v_i v_{i-j}^T\right] \tag{2.47}$$

The term $E\left[v_i \epsilon_{i-j}^T\right]$ in this equation is zero since future noise is uncorrelated with the past state. The term

$E[\epsilon_i \epsilon_{i-j}^T]$ can be evaluated as follows:

$$E\left[\epsilon_i \epsilon_{i-j}^T\right] = E\left[E[\epsilon_i x_{i-j}^T | Z_i] - E[\epsilon_i \bar{x}_{i-j}^T | Z_i]\right]$$

$$= E\left[E[\epsilon_i x_{i-j}^T | Z_i] - E[\epsilon_i | Z_i]\bar{x}_{i-j}^T\right]$$

$$= E\left[\epsilon_i x_{i-j}^T\right] \tag{2.48}$$

For i not equal to i-j, the term $E[v_i v_{i-j}^T]$ is zero. Therefore, the correlation function becomes

$$E\left[\nu_i \nu_{i-j}^T\right] = H_i E\left[\epsilon_i x_{i-j}^T\right] H_{i-j}^T + H_i E\left[\epsilon_i v_{i-j}^T\right]$$

$$= H_i E\left[\epsilon_i \left(H_{i-j} x_{i-j} + v_{i-j}\right)^T\right]$$

$$= H_i E\left[E\left[\epsilon_i z_{i-j}^T | Z_i\right]\right]$$

$$= H_i E\left[E(\epsilon_i | Z_i) z_{i-j}^T\right]$$

$$= 0 \tag{2.49}$$

This equation demonstrates that the innovation process is a white-noise process (i.e., the innovations are

independent from time point to time point). This outcome is the result to be expected for an optimum filter.

It is interesting to note that, while the innovations are statistically independent as shown above, they are related to the error of the predicted state error ϵ_i which can be defined by a recursive relationship. The equation relating the innovation and predicted state error is

$$\begin{aligned} \nu_i &= z_i - H_i\overline{x}_i \\ &= H_i(x_i - \overline{x}_i) + v_i \\ &= H_i\epsilon_i + v_i \end{aligned} \tag{2.50}$$

The predicted state error can be written

$$\begin{aligned} \epsilon_i &= x_i - \overline{x}_i \\ &= \phi_{i/i-1}x_{i-1} + \Gamma_{i/i-1}w_{i-1} - \phi_{i/i-1}\hat{x}_{i-1} \end{aligned} \tag{2.51}$$

Substitution for x_{i-1} gives the desired recursive relationship

$$\epsilon_i = \phi_{i/i-1}\left[I-K_{i-1}H_{i-1}\right](x_{i-1}-\bar{x}_{i-1})$$

$$+ \Gamma_{i/i-1}w_{i-1} - \phi_{i/i-1}K_{i-1}v_{i-1}$$

$$= \phi_{i/i-1}\left[I-K_{i-1}H_{i-1}\right]\epsilon_{i-1}$$

$$+ \Gamma_{i/i-1}w_{i-1} - \phi_{i/i-1}K_{i-1}v_{i-1} \qquad (2.52)$$

2.2 EXTENSION OF THE OPTIMUM STATE ESTIMATE

This section transforms the linear difference equations of a stochastic system with uncertain parameters to nonlinear difference equations. The extended Kalman filter is then derived to solve this nonlinear estimation problem.

2.2.1 Nonlinear System Equations

In the preceding section, the Kalman filter was derived for a stochastic system represented by linear difference equations. The derivation was based on the assumption that the parameters in the system equations were known. If these parameters are unknown, they may be treated as additional states to be estimated; however, the system equations become nonlinear.

Consider augmenting the state vector with the unknown parameters to form a new state vector x_i' given by

$$x_i' = \begin{bmatrix} x_i \\ \alpha \end{bmatrix} \tag{2.53}$$

The state and measurements may then be rewritten such that

$$x_i' = f(x_{i-1}', u_{i-1}) + g(x_{i-1}')w_{i-1} \tag{2.54}$$

$$z_i = h(x_i') + v_i \tag{2.55}$$

where $\phi_{i/i-1}(\alpha)x_{i-1} + \psi_{i/i-1}(\alpha)u_{i-1}$ is inherent in $f(x_{i-1}', u_{i-1})$ and $H_i(\alpha)x_i$ is represented by $h(x_i')$. These equations are, in general, nonlinear, and thus provide the basis for the derivation of the extended Kalman filter.

2.2.2 State Estimation for Nonlinear System Equations

Assume that the measurements Z_{i-1} are available so that the optimum state estimate at time i-1, $\hat{x}_{i-1}'$, and the predicted state estimate at time i, $\bar{x}_i'$, are provided. The state and measurement equations can then be expanded in a Taylor series about these estimates. Introducing the expansions truncated to zero or first order,

$$f(x'_{i-1},u_{i-1}) = f(\hat{x}'_{i-1},u_{i-1}) + \frac{\partial f}{\partial \hat{x}'_{i-1}} \left[x'_{i-1}-\hat{x}'_{i-1}\right] \tag{2.56}$$

$$g(x'_{i-1}) = g(\hat{x}'_{i-1}) \tag{2.57}$$

$$h(x'_i) = h(\bar{x}'_i) + \frac{\partial h}{\partial x'_i} (x'_i-\bar{x}'_i) \tag{2.58}$$

yields new system equations:

$$x'_i = \phi'_{i/i-1}x'_{i-1} + u'_{i-1} + \Gamma'_{i/i-1}w_{i-1} \tag{2.59}$$

$$z'_i = H'_i x'_i + v_{i-1} \tag{2.60}$$

where

$$\phi'_{i/i-1} = \frac{\partial f(\hat{x}'_{i-1},u_{i-1})}{\partial \hat{x}'_{i-1}} \tag{2.61}$$

$$u'_{i-1} = f(\hat{x}'_{i-1},u_{i-1}) - \frac{\partial f(\hat{x}_{i-1},u_{i-1})}{\partial x'_{i-1}} \hat{x}'_{i-1} \tag{2.62}$$

$$\Gamma'_{i/i-1} = g(\hat{x}'_{i-1}) \tag{2.63}$$

$$z_i' = z_i - h(\overline{x}') + \frac{\partial h(x_i')}{\partial \overline{x}_i'} \overline{x}_i' \tag{2.64}$$

$$H_i' = \frac{\partial h(\overline{x}_i')}{\partial \overline{x}_i'} \tag{2.65}$$

By comparison, these system equations are identical in form to the basic equations used to derive the Kalman filter (i.e., the equations are linear with known matrices and known inputs). Therefore, the estimate of x_i' can be developed analogously using the Kalman filter algorithms. The estimate of x_i' after i measurements is given by the equation

$$\hat{x}_i' = \overline{x}_i' + K_i'(z_i' - H_i'\overline{x}_i')$$

$$= \overline{x}_i' + K_i'\left[z_i - h(\overline{x}_i') + \frac{\partial h(\overline{x}_i')}{\partial \overline{x}_i'} \overline{x}_i' - \frac{\partial h(\overline{x}_i')}{\partial \overline{x}_i'} \overline{x}_i'\right]$$

$$= \overline{x}_i' + K_i'\left[z_i - h(\overline{x}_i')\right] \tag{2.66}$$

where the predicted value of x_i', given Z_{i-1} measurements, is

$$\overline{x}_i' = f(\hat{x}_{i-1}', u_{i-1}) \tag{2.67}$$

and the gain K_i' is

$$K_i' = P_{i/i-1}' H_i'^T \left[H_i' P_{i/i-1}' H_i'^T + R_i \right]^{-1}$$

$$= P_{i/i-1}' \frac{\partial h(\bar{x}_i')}{\partial x_i'}^T \left[\frac{\partial h(\bar{x}_i')}{\partial \bar{x}_i'} P_{i/i-1}' \frac{\partial h(\bar{x}_i')}{\partial \bar{x}_i'}^T + R_i \right]^{-1} \tag{2.68}$$

The covariance of the error in the predicted estimate $P_{i/i-1}'$ is

$$P_{i/i-1}' = \phi_{i/i-1}' P_{i-1}' \phi_{i/i-1}'^T + \Gamma_{i/i-1}' Q_{i-1} \Gamma_{i/i-1}'^T$$

$$= \frac{\partial f(\hat{x}_{i-1}', u_{i-1})}{\partial \hat{x}_{i-1}'} P_{i-1}' \frac{\partial f(\hat{x}_{i-1}', u_{i-1})}{\partial \hat{x}_{i-1}'}^T$$

$$+ \Gamma_{i/i-1}'(\hat{x}_{i-1}') Q_{i-1} \Gamma_{i/i-1}'^T(\hat{x}_{i-1}') \tag{2.69}$$

and the covariance of the error in state estimate P_i' is

$$P_i' = \left[I - K_i' H_i' \right] P_{i/i-1}' \left[I - K_i' H_i' \right]^T + K_i' R_i K_i'^T$$

$$= \left[I - K_i' \frac{\partial h(\bar{x}_i')}{\partial \bar{x}_i'} \right] P_{i/i-1}' \left[I - K_i' \frac{\partial h(\bar{x}_i')}{\partial \bar{x}_i'} \right]^T + K_i' R_i K_i'^T \tag{2.70}$$

Equations (2.66) and (2.70) provide the complete set of relationships for the extended Kalman filter. Estimation of both the state x_i and the parameters α is accomplished by this filter.

CHAPTER 3

MODIFIED KALMAN FILTER
FOR SYSTEMS WITH UNCERTAIN PARAMETERS

This chapter proposes a modification to the Kalman filter for state estimation of discrete-time systems when the parameters in the system equations are not known exactly. The modification includes the addition of a bias term to the filter and a change in the filter gain to reflect the uncertainty in the system parameters. The error covariance matrices of the filter are also amended as a result of the parameter uncertainties. The modification to the Kalman filter is based on the assumption that the parameter uncertainties are relatively small so that the terms containing the uncertain parameters can be expanded to first order in these parameters. Section 3.1 delineates this expansion and provides the equational relationships that are utilized in Section 3.2 to define the modified Kalman filter.

3.1 SYSTEM MODEL MODIFICATION

Parenthetically, it should be recognized that a fundamental assumption in this development is that a linear model has been defined which adequately represents the behavior of a physical system, but that certain parameters

in this linear description are not know exactly. Thus, equations of the model depict those aspects of a physical process that are most significant to a user's purposes. The filter consequently does not try to estimate the "true" values of the states of the system, but attempts to determine the values which, when substituted into the model, yield a model output behavior that best duplicates the actual system performance (which can be measured only with limited certainty). Moreover, this optimal replication is achieved for a single set of input and output sequences, with no guarantee that the same model provides an optimal system representation for other sequences as well.

The ability to perform the estimation will thus be determined by the conditions by which the mathematical model serves as an adequate system representation. If the estimation is impossible with the originally formulated problem, one may need to incorporate different measurements, additional measurements, or alter the system model, in order to satisfy these conditions.

However, given that system model stipulated in Chapter 1 is adequate and that the variances of the parameters $\sigma_1 \ldots \sigma_p$ are sufficiently small, the matrices containing the uncertain parameters can be expanded to first order about the nominal value of the parameters designated generically α_o. The transition, input, and noise input matrices become

$$\phi_{i/i-1}(\alpha) = \phi_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\phi_{i/i-1}\delta\alpha_k \tag{3.1}$$

$$\psi_{i/i-1}(\alpha) = \psi_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\psi_{i/i-1}\delta\alpha_k \tag{3.2}$$

$$\Gamma_{i/i-1}(\alpha) = \Gamma_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\Gamma_{i/i-1}\delta\alpha_k \tag{3.3}$$

where the lower left subscript α_k indicates differentiation with respect to the k^{th} parameter. Substituting into the state equation gives

$$x_i = \left[\phi_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\phi_{i/i-1}\delta\alpha_k\right]x_{i-1}$$

$$+\left[\psi_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\psi_{i/i-1}\delta\alpha_k\right]u_{i-1} \qquad \text{(cont)}$$

$$+\left[\Gamma_{i/i-1}(\alpha_o) + \sum_{k=1} {}_{\alpha_k}\Gamma_{i/i-1}\delta\alpha_k\right] w_{i-1} \tag{3.4}$$

Equation (3.4) may be separated into two components

$$x_i = x_i(\alpha_o) + \sum_{k=1} {}_{\alpha_k}x_i\delta\alpha_k \tag{3.5}$$

where

$$x_i(\alpha_o) = \phi_{i/i-1}(\alpha_o)x_{i-1} + \psi_{i/i-1}(\alpha_o)u_{i-1} + \Gamma_{i/i-1}(\alpha_o)w_{i-1} \tag{3.6}$$

$${}_{\alpha_k}x_i = {}_{\alpha_k}\phi_{i/i-1}x_{i-1} + {}_{\alpha_k}\psi_{i/i-1}u_{i-1} + {}_{\alpha_k}\Gamma_{i/i-1}w_{i-1} \tag{3.7}$$

and where, again, the lower left subscript indicates differentiation with respect to the k^{th} parameter α_k. The first term in Equation (3.5) defines the transition of the states and the effect of the inputs from time i-1 to time i based on nominal values for the parameters α_o. The second term provides the change in the states at time i due to the variation $\delta\alpha$ from the nominal value α_o.

The measurement matrix can also be expanded to first order and written

$$H_i(\alpha) = H_i(\alpha_o) + \sum_k {}_{\alpha_k}H_i \delta\alpha_k \qquad (3.8)$$

where the lower left subscript indicates differentiation with respect to the k^{th} parameter. The measurement equation is thus given by

$$z_i = \left[H_i(\alpha_o) + \sum_k {}_{\alpha_k}H_i \delta\alpha_k\right] x_i + v_i \qquad (3.9)$$

and utilizing Equation (3.4), becomes

$$z_i = H_i(\alpha_o)\left[\phi_{i/i-1}(\alpha_o)x_{i-1} + \psi_{i/i-1}(\alpha_o)u_{i-1} + \Gamma_{i/i-1}(\alpha_o)w_{i-1}\right]$$

$$+\sum_k \Bigg\{H_i(\alpha_o)\left[{}_{\alpha_k}\phi_{i/i-1}x_{i-1} + {}_{\alpha_k}\psi_{i/i-1}u_{i-1} + {}_{\alpha_k}\Gamma_{i/i-1}w_{i-1}\right]$$

$$+{}_{\alpha_k}H_i\left[\phi_{i/i-1}(\alpha_o)x_{i-1} + \psi_{i/i-1}(\alpha_o)u_{i-1} + \Gamma_{i/i-1}(\alpha_o)w_{i-1}\right]\Bigg\}\delta\alpha_k$$

$$+\sum_{\ell}\sum_k \left\{{}_{\alpha_\ell}H_i\left[{}_{\alpha_k}\phi_{i/i-1}x_{i-1} + {}_{\alpha_k}\psi_{i/i-1}u_{i-1} + {}_{\alpha_k}\Gamma_{i/i-1}w_{i-1}\right]\right\}\delta\alpha_\ell\delta\alpha_k$$

$$+ v_i \qquad (3.10)$$

This equation and the other equations above provide new expressions for the system model, which enable modification of the Kalman filter and permit derivation of a state estimator for stochastic systems with uncertain parameters.

3.2 DERIVATION OF THE MODIFIED KALMAN FILTER

The Kalman filter is an optimum estimator for linear stochastic systems only when the parameters of the system are known. When the system parameters deviate from assigned values, the decreased accuracy in the estimate computed by the Kalman filter may be unacceptable. If modifications to the Kalman filter are considered, an improved estimate can be made which may be sufficiently accurate.

How shall the Kalman filter be modified? The most likely values for the parameters are the expected values and, thus, terms containing these parameters should be determined with these nominal values. However, since the true parameter values may differ from the nominal values, the modified filter may be biased. Therefore, a term should be added to the filter equation to compensate for this bias. In addition, the filter gain should be derived to reflect the uncertainty in the system parameters so that a proper weighting can be given in the filter to the measurements.

The form of the modified filter is thus given by the equation

$$\hat{x}_i = \overline{x}_i + G_i\left[z_i - H_i\overline{x}_i\right] + \mu_i \tag{3.11}$$

where

$$\overline{x}_i = \phi_{i/i-1}\hat{x}_{i-1} + \psi_{i/i-1}u_{i-1} \tag{3.12}$$

and

$\hat{x}_i$ - filter estimate at time i after Z_i measurements

$\overline{x}_i$ - predicted state at time i after Z_{i-1} measurements

z_i - measurement vector at time i

u_{i-1} - input vector at time i-1

G_i - filter gain modified to reflect parameter uncertainty

μ_i - filter bias

$H_i, \phi_{i/i-1}, \psi_{i/i-1}$ - matrix relationships evaluated with α_o

α_o - expected value of parameters α

The bias term μ_i and the gain G_i in the filter equation are derived in the following subsections. The covariance

of the filter error is also determined. In addition, the complete set of algorithms is presented and a limited discussion of the filter stability is included in the next section.

3.2.1 Filter Bias

For the filter to be unbiased, the expected value of the difference between the state and the filter estimate e_i must be zero for all time (i.e., $E[e_i] = 0$ for i = 1, 2, 3,...). Utilizing Equations (3.4), (3.10), and (3.11), this difference is written

$$
\begin{aligned}
e_i &= x_i - \hat{x}_i \\
&= \left[I - G_i H_i\right] \phi_{i/i-1} e_{i-1} \\
&\quad + \left[I - G_i H_i\right] \Gamma_{i/i-1} w_{i-1} - G_i v_i - \mu_i \\
&\quad + \sum_k \Big\{ \alpha_k \left[\phi_{i/i-1} - G_i H_i \phi_{i/i-1}\right] x_{i-1} \\
&\quad + \alpha_k \left[\psi_{i/i-1} - G_i H_i \psi_{i/i-1}\right] u_{i-1} \\
&\quad + \alpha_k \left[\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}\right] w_{i-1} \Big\} \delta\alpha_k
\end{aligned} \tag{3.13}
$$

where

$${}_{\alpha_k}\left[\phi_{i/i-1} - G_i H_i \phi_{i/i-1}\right]$$

$$= {}_{\alpha_k}\phi_{i/i-1} - G_{i\alpha_k} H_i \phi_{i/i-1} - G_i H_{i\alpha_k} \phi_{i/i-1} \qquad (3.14)$$

$${}_{\alpha_k}\left[\psi_{i/i-1} - G_i H_i \phi_{i/i-1}\right]$$

$$= {}_{\alpha_k}\psi_{i/i-1} - G_{i\alpha_k} H_i \psi_{i/i-1} - G_i H_{i\alpha_k} \psi_{i/i-1} \qquad (3.15)$$

$${}_{\alpha_k}\left[\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}\right]$$

$$= {}_{\alpha_k}\Gamma_{i/i-1} - G_{i\alpha_k} H_i \Gamma_{i/i-1} - G_i H_{i\alpha_k} \Gamma_{i/i-1} \qquad (3.16)$$

and where for simplicity, $\phi_{i/i-1}(\alpha_o)$, $\psi_{i/i-1}(\alpha_o)$ and $\Gamma_{i/i-1}(\alpha_o)$ have been shortened to $\phi_{i/i-1}$, $\psi_{i/i-1}$, and $\Gamma_{i/i-1}$, respectively, and the second order terms of $\delta\alpha$ have been dropped. Taking the expected value of Equation (3.13) and recognizing that the noise inputs are zero-mean processes and are uncorrelated with the parameter variations gives

$$E[e_i] = (I - G_iH_i)\phi_{i/i-1}E[e_{i-1}]$$

$$+ (I - G_iH_i)\Gamma_{i/i-1}E[w_{i-1}] - G_iE[v_i] - E[\mu_i]$$

$$+\sum_k \Big\{ \alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})\ E[x_{i-1}\delta\alpha_k]$$

$$+ \alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})\ E[u_{i-1}\delta\alpha_k]$$

$$+ \alpha_k(\Gamma_{i/i-1} - G_iH_i\Gamma_{i/i-1})\ E[w_{i-1}\delta\alpha_k]$$

$$= -\mu_i + \sum_k \alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})\ E[x_{i-1}\delta\alpha_k] \qquad (3.17)$$

If the term $E[x_{i-1}\delta\alpha_k]$ is approximated by

$$E[x_{i-1}\delta\alpha] = 0 \qquad (3.18)$$

The bias term of the filter is then given by

$$\mu_i = 0 \qquad (3.19)$$

The filter error can thus be reduced to

$$e_i = (I - G_iH_i)\phi_{i/i-1}e_{i-1} + (I - G_iH_i)\Gamma_{i/i-1}w_{i-1} - G_iv_i$$

$$+\sum_k \Big\{ \alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})x_{i-1}$$

$$+ \alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})u_{i-1}$$

$$+ \alpha_k(\Gamma_{i/i-1} - G_iH_i\Gamma_{i/i-1})w_{i-1}\Big\}\delta\alpha_k \tag{3.20}$$

This expression can now be used to define the filter gain.

3.2.2 Filter Gain

The Kalman filter is a minimum error variance filter (see Appendix B). It is therefore desirable to achieve this property in a modified Kalman filter. If the gain in the modified filter is selected so as to minimize the expected value of a quadratic function of the filter, this objective can be achieved. The equation which states this objective is

$$\frac{\partial E\left[e_i^T e_i\right]}{\partial G_i} = 0 \tag{3.21}$$

or, alternately,

$$\frac{\partial \mathrm{tr}\ E[e_i e_i^T]}{\partial G_i} = 0 \tag{3.22}$$

where tr is the matrix trace.

Pursuing the above objective, the covariance equation must first be derived. Using Equation (3.20) and recognizing that the parameter variations are uncorrelated (i.e., $E[\delta\alpha_k \delta\alpha_\ell] = 0$), the covariance may be written

$$E[e_i e_i^T] = (I-G_iH_i)\phi_{i/i-1}E[e_{i-1}e_{i-1}^T]\phi_{i/i-1}^T(I-G_iH_i)^T$$

$$+ (I-G_iH_i)\Gamma_{i/i-1}E[w_{i-1}w_{i-1}^T]\Gamma_{i/i-1}^T(I-G_iH_i)^T + G_iE[v_iv_i^T]G_i^T$$

$$+ (I-G_iH_i)\phi_{i/i-1}E[e_{i-1}w_{i-1}^T]\Gamma_{i/i-1}^T(I-G_iH_i)^T$$

$$+ (I-G_iH_i)\Gamma_{i/i-1}E[w_{i-1}e_{i-1}^T]\phi_{i/i-1}^T(I_i-G_iH_i)^T$$

$$- (I-G_iH_i)\phi_{i/i-1}E[e_{i-1}v_i^T]G_i^T - G_iE[v_ie_{i-1}^T]\phi_{i/i-1}^T(I-G_iH_i)^T$$

$$- (I-G_iH_i)\Gamma_{i/i-1}E[w_{i-1}v_i^T]G_i^T - G_iE[v_iw_{i-1}^T]\Gamma_{i/i-1}^T(I-G_iH_i)^T$$

(cont)

$$
\begin{aligned}
&+ \sum_{k} \Big\{ (\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[e_{i-1} x_{i-1}^T \delta\alpha_k]_{\alpha_k} (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[x_{i-1} e_{i-1}^T \delta\alpha_k] (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \\
&+ (\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[e_{i-1} u_{i-1}^T \delta\alpha_k]_{\alpha_k} (\psi_{i/i-1} - G_i H_i \psi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\psi_{i/i-1} - G_i H_i \psi_{i/i-1}) E[u_{i-1} e_{i-1}^T \delta\alpha_k] (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \\
&+ (\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[e_{i-1} w_{i-1}^T \delta\alpha_k]_{\alpha_k} (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T \\
&+ {}_{\alpha_k}(\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} e_{i-1}^T \delta\alpha_k] (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \\
&+ (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} x_{i-1}^T \delta\alpha_k]_{\alpha_k} (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T E[x_{i-1} w_{i-1}^T \delta\alpha_k] (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T \\
&+ (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} u_{i-1}^T \delta\alpha_k]_{\alpha_k} (\psi_{i/i-1} - G_i H_i \psi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\psi_{i/i-1} - G_i H_i \psi_{i/i-1}) E[u_{i-1} w_{i-1}^T \delta\alpha_k] (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T \\
&+ (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} w_{i-1}^T \delta\alpha_k]_{\alpha_k} (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T
\end{aligned}
$$

(cont)

$$+ {}_{\alpha_k}(\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} w_{i-1}^T \delta\alpha_k] (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T$$

$$- G_i E[v_i x_{i-1}^T \delta\alpha_k]_{\alpha_k} (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T$$

$$- {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[x_{i-1} v_i^T \delta\alpha_k] G_i^T$$

$$- G_i E[v_i u_{i-1}^T \delta\alpha_k]_{\alpha_k} (\psi_{i/i-1} - G_i H_i \psi_{i/i-1})^T$$

$$- {}_{\alpha_k}(\psi_{i/i-1} - G_i H_i \psi_{i/i-1}) E[u_{i-1} v_{i-1}^T \delta\alpha_k] G_i^T$$

$$- G_i E[v_i w_{i-1}^T \delta\alpha_k]_{\alpha_k} (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T$$

$$- {}_{\alpha_k}(\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} v_i^T \delta\alpha_k] G_i^T \Big\}$$

$$+ \sum_k {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[x_{i-1} x_{i-1}^T \delta\alpha_k^2]_{\alpha_k} (\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T$$

$$+ {}_{\alpha_k}(\psi_{i/i-1} - G_i H_i \psi_{i/i-1}) E[u_{i-1} u_{i-1}^T \delta\alpha_k^2]_{\alpha_k} (\psi_{i/i-1} - G_i H_i \psi_{i/i-1})^T$$

$$+ {}_{\alpha_k}(\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1}) E[w_{i-1} w_{i-1}^T \delta\alpha_k^2]_{\alpha_k} (\Gamma_{i/i-1} - G_i H_i \Gamma_{i/i-1})^T$$

$$+ {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1}) E[x_{i-1} u_{i-1}^T \delta\alpha_k^2]_{\alpha_k} (\psi_{i/i-1} - G_i H_i \psi_{i/i-1})^T$$

(cont)

$$+ \alpha_k(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})E[u_{i-1}x^T_{i-1}\delta\alpha^2_k]_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T$$

$$+ \alpha_k(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})E[x_{i-1}w^T_{i-1}\delta\alpha^2_k]_{\alpha_k}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T$$

$$+ \alpha_k(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})E[w_{i-1}x^T_{i-1}\delta\alpha^2_k]_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T$$

$$+ \alpha_k(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})E[u_{i-1}w^T_{i-1}\delta\alpha^2_k]_{\alpha_k}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T$$

$$+ \alpha_k(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})E[w_{i-1}u^T_{i-1}\delta\alpha^2_k]_{\alpha_k}(\psi_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T \tag{3.23}$$

The expected operations in this equation must now be evaluated. The term $E[e_{i-1}e^T_{i-1}]$ is simply the filter error covariance at time i-1, which will be designated P_{i-1}. The noise inputs w_{i-1} and v_i, as previously defined, are uncorrelated with covariances Q_{i-1} and R_i, respectively. Terms involving single functions of the noise inputs are zero since the terms are uncorrelated with the noise inputs and the expected values of the noise inputs are zero. An example of this result is

$$E[w_{i-1}x^T_{i-1}] = E[w_{i-1}]E[x_{i-1}]^T$$

$$= 0 \cdot E[x_{i-1}]$$

$$= 0 \qquad (3.24)$$

Terms involving the noise at time i with results at time i-1 are also zero since future noise is uncorrelated with past results. Remaining terms containing the parameter variation $\delta\alpha_k$ are set to zero. Other terms containing $\delta\alpha_k$ are estimated using the variance σ_k. This evaluation and the others above are summarized in the following equations.

$$E[e_{i-1}e^T_{i-1}] = P_{i-1} \qquad (3.25)$$

$$E[w_{i-1}w^T_{i-1}] = Q_{i-1} \qquad (3.26)$$

$$E[v_i v^T_i] = R_i \qquad (3.27)$$

$$E[e_{i-1}w^T_{i-1}] = E[w_{i-1}e^T_{i-1}] = 0 \qquad (3.28)$$

$$E[e_{i-1}v^T_i] = E[v_i e^T_{i-1}] = 0 \qquad (3.29)$$

$$E[w_{i-1}v_i^T] = E[v_i w_{i-1}^T] = 0 \tag{3.30}$$

$$E[e_{i-1}x_{i-1}^T\delta\alpha_k] = E[x_{i-1}e_{i-1}^T\delta\alpha_k] = 0 \tag{3.31}$$

$$E[e_{i-1}u_{i-1}^T\delta\alpha_k] = E[e_{i-1}u_{i-1}^T\delta\alpha_k] = 0 \tag{3.32}$$

$$E[e_{i-1}w_{i-1}^T\delta\alpha_k] = E[w_{i-1}e_{i-1}^T\delta\alpha_k] = 0 \tag{3.33}$$

$$E[w_{i-1}x_{i-1}^T\delta\alpha_k] = E[x_{i-1}w_{i-1}^T\delta\alpha_k] = 0 \tag{3.34}$$

$$E[w_{i-1}u_{i-1}^T\delta\alpha_k] = E[u_{i-1}w_{i-1}^T\delta\alpha_k] = 0 \tag{3.35}$$

$$E[w_{i-1}w_{i-1}^T\delta\alpha_k] = 0 \tag{3.36}$$

$$E[v_i x_{i-1}^T\delta\alpha_k] = E[x_{i-1}v_i^T\delta\alpha_k] = 0 \tag{3.37}$$

$$E[v_i u_{i-1}^T\delta\alpha_k] = E[u_{i-1}v_i^T\delta\alpha_k] = 0 \tag{3.38}$$

$$E[v_i w_{i-1}^T\delta\alpha_k] = E[w_{i-1}v_i^T\delta\alpha_k] = 0 \tag{3.39}$$

$$E[x_{i-1}x_{i-1}\delta\alpha_k^2] \approx \sigma_k(P_{i-1} + \hat{x}_{i-1}\hat{x}_{i-1}^T) \tag{3.40}$$

$$E[u_{i-1}u_{i-1}^T\delta\alpha_k^2] = \sigma_k[u_{i-1}u_{i-1}^T] \tag{3.41}$$

$$E[w_{i-1}w^T_{i-1}\delta\alpha^2_k] = \sigma_k\, Q_{i-1} \tag{3.42}$$

$$E[x_{i-1}u^T_{i-1}\delta\alpha^2_k] \approx \sigma_k\, \hat{x}_{i-1}u^T_{i-1} \tag{3.43}$$

$$E[u_{i-1}x^T_{i-1}\delta\alpha^2_k] \approx \sigma_k\, u_{i-1}\hat{x}^T_{i-1} \tag{3.44}$$

$$E[x_{i-1}w^T_{i-1}\delta\alpha^2_k] = E[w_{i-1}x^T_{i-1}\delta\alpha^2_k] = 0 \tag{3.45}$$

$$E[u_{i-1}w^T_{i-1}\delta\alpha^2_k] = E[w_{i-1}u^T_{i-1}\delta\alpha^2_k] = 0 \tag{3.46}$$

The covariance then becomes

$$E[e_ie^T_i] = (I-G_iH_i)\phi_{i/i-1}P_{i-1}\phi^T_{i/i-1}(I-G_iH_i)^T$$

$$+ (I-G_iH_i)\Gamma_{i/i-1}Q_{i-1}\Gamma^T_{i/i-1}(I-G_iH_i)^T + G_iR_iG^T_i$$

$$+\sum_k \sigma_k \Big\{ {}_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i-1})(P_{i-1}+\hat{x}_{i-1}\hat{x}^T_{i-1})_{\alpha_k}(\phi_{i-1}-G_iH_i\phi_{i-1})^T$$

$$+ {}_{\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})u_{i-1}u^T_{i-1\,\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})^T$$

$$+ {}_{\alpha_k}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})Q_{i-1\,\alpha_k}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T$$

(cont)

$$+ \alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})\hat{x}_{i-1}u^T_{i-1}\alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})^T$$

$$+ \alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})u_{i-1}\hat{x}^T_{i-1}\alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})^T \quad (3.47)$$

Minimizing the trace of this equation with respect to the gain G_i and recognizing the symmetry of matrices within some of the individual terms gives the matrix equation

$$- 2(I - G_iH_i)\phi_{i/i-1}P_{i-1}\phi^T_{i/i-1}H^T_i$$

$$- 2(I - G_iH_i)\Gamma_{i/i-1}Q_{i-1}\Gamma^T_{i/i-1}H^T_i + 2G_iR_i$$

$$- 2\sum_k \sigma_k \Big[\alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})(P_{i-1} + \hat{x}_{i-1}\hat{x}^T_{i-1})\alpha_k(H_i\phi_{i/i-1})^T$$

$$+ \alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})u_{i-1}u^T_{i-1}\alpha_k(H_i\psi_{i/i-1})^T$$

$$+ \alpha_k(\Gamma_{i/i-1} - G_iH_i\Gamma_{i/i-1})Q_{i-1}\alpha_k(H_i\Gamma_{i/i-1})^T$$

$$+ \alpha_k(\phi_{i/i-1} - G_iH_i\phi_{i/i-1})\hat{x}_{i-1}u^T_{i-1}\alpha_k(H_i\psi_{i/i-1})^T$$

$$+ \alpha_k(\psi_{i/i-1} - G_iH_i\psi_{i/i-1})u_{i-1}\hat{x}^T_{i-1}\alpha_k(H_i\phi_{i/i-1})^T$$

$$= 0 \quad (3.48)$$

where

$$ {}_{\alpha_k}(H_i\phi_{i/i-1}) = {}_{\alpha_k}H_i\phi_{i/i-1} + H_{i\alpha_k}\phi_{i/i-1} \tag{3.49} $$

$$ {}_{\alpha_k}(H_i\psi_{i/i-1}) = {}_{\alpha_k}H_i\psi_{i/i-1} + H_{i\alpha_k}\psi_{i/i-1} \tag{3.50} $$

$$ {}_{\alpha_k}(H_i\Gamma_{i/i-1}) = {}_{\alpha_k}H_i\Gamma_{i/i-1} + H_{i\alpha_k}\Gamma_{i/i-1} \tag{3.51} $$

The modified gain is therefore

$$ G_i = \left\{\phi_{i/i-1}P_{i-1}(H_i\phi_{i/i-1})^T + \Gamma_{i/i-1}Q_{i-1}(H_i\Gamma_{i/i-1})^T + \sum_k {}_{\alpha_k}r_i\right\} $$

$$ \left\{(H_i\phi_{i/i-1})P_{i-1}(H_i\phi_{i/i-1})^T + (H_i\Gamma_{i/i-1})Q_{i-1}(H_i\Gamma_{i/i-1})^T\right. $$

$$ \left. + R_i + \sum_k (H_{i\alpha_k}r_i + {}_{\alpha_k}H_{i\alpha_k}s_i)\right\}^{-1} \tag{3.52} $$

where

$$ {}_{\alpha_k}r_i = \sigma_k\left\{{}_{\alpha_k}\phi_{i/i-1}(P_{i-1}+\hat{x}_{i-1}\hat{x}^T_{i-1})\,{}_{\alpha_k}(H_i\phi_{i/i-1})^T\right. $$

$$ + {}_{\alpha_k}\psi_{i/i-1}u_{i-1}u^T_{i-1}\,{}_{\alpha_k}(H_i\psi_{i/i-1})^T + {}_{\alpha_k}\Gamma_{i/i-1}Q_{i-1}\,{}_{\alpha_k}(H_i\Gamma_{i/i-1})^T $$

(cont)

$$+ \alpha_k \phi_{i/i-1} \hat{x}_{i-1} u_{i-1}^T \alpha_k (H_i \psi_{i/i-1})^T$$

$$\left. + \alpha_k \psi_{i/i-1} u_i \hat{x}_{i-1}^T \alpha_k (H_i \phi_{i/i-1})^T \right\} \tag{3.53}$$

$$\alpha_k s_i = \sigma_k \Big\{ \phi_{i/i-1} (P_{i-1} + \hat{x}_{i-1} \hat{x}_{i-1}^T) \alpha_k (H_i \phi_{i/i-1})^T$$

$$+ \psi_{i/i-1} u_{i-1} u_{i-1}^T \alpha_k (H_i \psi_{i/i-1})^T + \Gamma_{i/i-1} Q_{i-1} \alpha_k (H_i \Gamma_{i/i-1})^T$$

$$\left. + \phi_{i/i-1} \hat{x}_{i-1} u_{i-1}^T \alpha_k (H_i \psi_{i/i-1})^T + \psi_{i/i-1} u_{i-1} \hat{x}_{i-1}^T \alpha_k (H_i \phi_{i/i-1})^T \right\} \tag{3.54}$$

This gain equation contains terms found in the Kalman gain equation and, in fact, is identical to the Kalman gain when the parameter variations are zero. Most of the additional terms in the expression are similar in form to the nominal terms of the Kalman gain. The exception is the quadratic term of the input u_{i-1} and the cross products terms of the state x_{i-1} and input u_{i-1}. These terms occur as a result of the uncertainty of the input matrix $\psi_{i/i-1}$.

3.2.3 Covariance of the Filter Error

The covariance of the filter error at time P_i is defined by

$$P_i = E\left[(x_i-\hat{x}_i)(x_i-\hat{x}_i)^T\right] = E\left[e_i e_i^T\right] \tag{3.55}$$

Using Equation (3.47) gives

$$P_i = (\phi_{i/i-1}-G_iH_i\phi_{i/i-1})P_{i-1}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T$$

$$+ (\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})Q_{i-1}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T$$

$$+ G_iR_iG_i + \sum_k \alpha_k t_i \tag{3.56}$$

where

$$\alpha_k t_i =$$

$$\sigma_k\Big\{\alpha_k(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})(P_{i-1}+\hat{x}_{i-1}\hat{x}^T_{i-1})\alpha_k(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T$$

$$+ \alpha_k(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})u_{i-1}u^T_{i-1}\alpha_k(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})^T$$

$$+ \alpha_k(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})Q_{i-1}\alpha_k(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T$$

$$+ \alpha_k(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})\hat{x}_{i-1}u^T_{i-1}\alpha_k(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})^T$$

(cont)

$$+ \alpha_k(\psi_{i/i-1} - G_i H_i \psi_{i/i-1}) u_{i-1} \hat{x}^T_{i-1} {}_{\alpha_k}(\phi_{i/i-1} - G_i H_i \phi_{i/i-1})^T \Big\} \tag{3.57}$$

The first three terms in this equation correspond to the error covariance of the Kalman filter. The additional terms reflect the increased uncertainty in the estimate due to parameter uncertainties in the system model. Some of these added terms might be envisioned if an expansion of the Kalman filter error covariance is considered. However, since the system model of the Kalman filter assumes a deterministic input u_i transmitted through a known input matrix $\psi_{i/i-1}(\alpha)$, terms involving the input are not contained in the Kalman filter error covariance (i.e., there is no uncertainty in known quantities) and, therefore, would not be anticipated in the expansion. Such terms do exist because of the parameter uncertainties within the input matrix $\psi_{i/i-1}(\alpha)$.

3.2.4 Filter Algorithms

The filter algorithms for state estimation are summarized here. The first step for time i-1 to time i is the computation of the gain

$$G_i = J_i(H_i J_i + R_i + S_i)^{-1} \tag{3.58}$$

where

$$J_i = P_{i/i-1}H_i^T + \sum_k \alpha_k r_i \tag{3.59}$$

$$S_i = \sum_k \alpha_k H_{i\alpha_k} s_i \tag{3.60}$$

with

$$P_{i/i-1} = \phi_{i/i-1}P_{i-1}\phi_{i/i-1}^T + \Gamma_{i/i-1}Q_{i-1}\Gamma_{i/i-1}^T \tag{3.61}$$

and

$$\begin{aligned}
{}_{\alpha_k}r_i = \sigma_k \Big\{ & {}_{\alpha_k}\phi_{i/i-1}(P_{i-1}+\hat{x}_{i-1}\hat{x}_{i-1}^T)_{\alpha_k}(H_i\phi_{i/i-1})^T \\
& + {}_{\alpha_k}\psi_{i/i-1}u_{i-1}u_{i-1}^T{}_{\alpha_k}(H_i\psi_{i/i-1})^T + {}_{\alpha_k}\Gamma_{i/i-1}Q_{i-1\alpha_k}(H_i\Gamma_{i/i-1})^T \\
& + {}_{\alpha_k}\phi_{i/i-1}\hat{x}_{i-1}u_{i-1\alpha_k}^T(H_i\psi_{i/i-1})^T + {}_{\alpha_k}\psi_{i/i-1}u_{i-1}\hat{x}_{i-1\alpha_k}^T(H_i\phi_{i/i-1})^T \Big\}
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
{}_{\alpha_k}s_i = \sigma_k \Big\{ & \phi_{i/i-1}(P_{i-1}+\hat{x}_{i-1}\hat{x}_{i-1}^T)_{\alpha_k}(H_i\phi_{i/i-1})^T \\
& + \psi_{i/i-1}u_{i-1}u_{i-1\alpha_k}^T(H_i\psi_{i/i-1})^T + \Gamma_{i/i-1}Q_{i-1\alpha_k}(H_i\Gamma_{i/i-1})^T
\end{aligned}$$

(cont)

$$+ \phi_{i/i-1}\hat{x}_{i-1}u^T_{i-1}\alpha_k(H_i\psi_{i/i-1})^T + \psi_{i/i-1}u_{i-1}\hat{x}^T_{i-1}\alpha_k(H_i\phi_{i/i-1})^T\Big\} \tag{3.54}$$

The bias term is zero

$$\mu_i = 0 \tag{3.19}$$

The filter estimate is then determined similarly to the Kalman filter expression in Equation (2.27) by the relationship

$$\hat{x}_i = (I - G_iH_i)\ \overline{x} + G_iz_i \tag{3.62}$$

where

$$\overline{x}_i = \phi_{i/i-1}\hat{x}_{i-1} + \psi_{i/i-1}u_{i-1} \tag{3.63}$$

Finally, the error covariance matrix is computer from the equation

$$P_i = (I - G_iH_i)P_{i/i-1}(I - G_iH_i)^T + G_iR_iG_i^T + T_i \tag{3.64}$$

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where

$$T_i = \sum_k \alpha_k t_i \tag{3.65}$$

and

$${}_{\alpha_k}t_i =$$

$$\begin{aligned}
&\sigma_k\Big\{{}_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})(P_{i-1}+\hat{x}_{i-1}\hat{x}^T_{i-1})\,{}_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})u_{i-1}u^T_{i-1}\,{}_{\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\Gamma_{i/i-1}\ G_iH_i\Gamma_{i/i-1})Q_{i-1}\,{}_{\alpha_k}(\Gamma_{i/i-1}-G_iH_i\Gamma_{i/i-1})^T \\
&+ {}_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})\hat{x}_{i-1}u^T_{i-1}\,{}_{\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})^T \\
&+ {}_{\alpha_k}(\psi_{i/i-1}-G_iH_i\psi_{i/i-1})u_{i-1}\hat{x}^T_{i-1}\,{}_{\alpha_k}(\phi_{i/i-1}-G_iH_i\phi_{i/i-1})^T\Big\}
\end{aligned} \tag{3.57}$$

Note that all matrices in the expressions above containing the parameters α are defined assuming the nominal parameter values α_o.

The advantage expected for this filter over the Kalman filter is the higher accuracy, and over the extended Kalman filter the advantage is the reduced computation burden. The modified filter requires only n equations for state estimation and n(n+1)/2 equations for computation of the error covariance. With p parameters in the system equations, the extended Kalman filter requires n+p equations for state estimation and (n+p)(n+p+1)/2 equations for the error covariance computation.

The gain in the modified filter includes additional terms relative to the standard Kalman gain. In general, when the parameter uncertainties exist in the state equation, the effect of these terms is to increase the gain since the measurement z_i is a more accurate assessment of the state than the prediction of the state given by $\bar{x}_i$. Conversely, when the parameter uncertainties exist in the measurement equation, the gain is decreased since the predicted state $\bar{x}_i$ is a more accurate appraisal than that derived from the measurement z_i.

3.3 <u>FILTER STABILITY</u>

In this subsection, the asymptotic property of the modified Kalman filter error is considered. The approach utilized in the assessment has been applied to the Kalman

filter. However, very little has been proven or is known about this property for the Kalman filter. For this reason and because of the difficulty in developing general stability results, the treatment here is limited.

The assessment makes use of a stability theorem commonly defined as the second, or direct, method of Liapunov. This theorem can be stated in the following manner. Consider a difference equation given by

$$y_i = A\, y_{i-1} \tag{3.66}$$

where A is a constant matrix, and let L be a positive definite Liapunov function of y so that the difference ΔL is

$$\Delta L = L(y_i) - L(y_{i-1}) \tag{3.67}$$

Now, if ΔL is negative semidefinite (definite), then the difference equation is stable (asymptotically stable); that is, y_i remains less than some small value as time approaches infinite. Conversely, if ΔL is positive definite, the difference equation is unstable.

The filter error as defined in Equation (3.20) consists of a homogeneous part followed by terms involving the noise inputs and perturbations of the state and input. Since the terms involving the noise are, on the average, zero, they can be ignored. In addition, if the state and input remain finite, these perturbation terms can be ignored. The remaining term is the homogeneous part which can be the difference equation used in the evaluation if the modified gain matrix G_i, the measurement matrix H_i, and state transition matrix $\phi_{i/i-1}$ are assumed to be constant matrices G, H, and ϕ, respectively. There are many problems in which the state transition and measurement matrices are constant, and this is not a major restriction. However, the implication of a constant gain matrix is that the filter error covariance is constant (see Section 3.2.4). In many problems, the covariance would not remain constant and, in fact, might be increasing in value with increasing time. Therefore, this evaluation is limited to determining values of the gain or other conditions which indicate stability.

Therefore, let the difference equation to be evaluated be given by

$$y_i = (I - GH)\phi y_{i-1} \tag{3.68}$$

and define the Liapunov testing function as

$$L = y_i^T y_i \quad . \tag{3.69}$$

which is clearly positive definite. The derivative of the Liapunov function is

$$\Delta L = y_i^T y_i - y_{i-1}^T y_{i-1}$$

$$= y_{i-1}^T \left[\phi^T (I - GH)^T (I - GH)\phi - I\right] y_{i-1} \tag{3.70}$$

If the term $\phi^T (I - GH)^T (I - GH)\phi - I$ is negative semidefinite, then the filter estimate is Liapunov stable. Equationally, this is

$$\phi^T (I - GH)^T (I - GH)\phi - I \leq 0 \tag{3.71}$$

The modified gain can assume a range of values. Conceptually, the lowest value is 0 when the measurements are disregarded. In this case, the filter is stable if

$$\phi^T \phi - I \leq 0 \tag{3.72}$$

For the scaler case, this is equivalent to

$$\phi^2 - 1 \leq 0 \tag{3.73}$$

or

$$-1 \leq \phi \leq 1 \tag{3.74}$$

At the other extreme, the measurements may determine the state exactly (i.e., R_i=0 and $\alpha_k H_i$=0 for all i and k). In this case, the gain G can be defined by (see Appendix C)

$$GH = I \tag{3.75}$$

The derivative of the Liapunov function then becomes

$$\Delta L = y_{i-1}^T \left[\phi^T (I-I)^T (I-I)\phi - I\right] y_{i-1}$$

$$= y_{i-1}^T (-I) y_{i-1} \tag{3.76}$$

which is unequivocally negative definite, and thus the filter estimate is stable.

With these equations, the basis is provided for developing limited guidelines for stability with the modified Kalman filter. In general, the filter is stable if

$$\phi^T(I - GH)^T(I - GH)\phi - I \leq 0 \tag{3.71}$$

Specifically, the filter would be expected to be stable if the state is asymptotically decreasing, which is represented by the expression

$$\phi^T\phi - I \leq 0 \tag{3.72}$$

Additionally, the filter is expected to be stable if the measurements are extremely accurate. This occurs when GH is approximately equal to I. For other conditions, the actual error may increase with time. However, the error covariance would also be expected to increase. The filter error is Gaussianly distributed and, therefore, the probability of any such error should be adequately determined with the filter error covariance.

CHAPTER 4

ADAPTIVE ESTIMATION FOR SYSTEMS WITH UNCERTAIN PARAMETERS

The Kalman filter derived in Chapter 2 is an optimum filter for stochastic systems with known parameters. In addition, the innovation sequence generated by the Kalman filter (i.e., the sequence of differences between the actual measurements and the predicted measurements) is a zero-mean Gaussian, white-noise process. When the parameters in the system are not known exactly, however, the filter is sub-optimum, the innovation sequence is altered, and adaptive action is required to correct the filter performance. In this chapter, this philosophy is applied to state estimation for systems with uncertain parameters. An adaptive maximum likelihood Bayesian estimator of the uncertain parameters is derived that functions in conjunction with a Kalman filter. The Kalman filter algorithms operate with the current estimate of the parameters to generate the innovations and other data utilized by the Bayesian estimator to improve the parameter estimates. With the new values for the parameters, the Kalman filter algorithms are again operated to regenerate the state estimates. Section 4.1 presents the basis for

the parameter estimation and derives a nonlinear equation for the parameter estimate. The solution of this nonlinear equation is then developed in Section 4.2. The adaptive filter algorithms are summarized in Section 4.3, and the convergence of the estimate is evaluated in Section 4.4.

4.1 PARAMETER ESTIMATION

This section develops a Bayesian parameter estimate from the variational terms of the conditional probability density function of the parameters, given the measurements. The estimate is equivalent to a maximum a posteriori estimator known also as the maximum likelihood Bayesian estimator. The expected value of the variational terms of the conditional density function is also presented.

4.1.1 Maximum a Posteriori Estimate from $-\ln p(\alpha|Z_N)$

One of the most useful Bayesian estimators is the maximum a posteriori estimator which is obtained by maximizing, with respect to the parameters, the conditional density function of the parameters $p(\alpha|Z_N)$ where Z_N is the vector of realized values of N measurements $z_1 \ldots z_N$. Rather than directly maximizing the conditional density function, however, it is more convenient to maximize the negative natural logarithm of the conditional density function. This is a valid alternative since, for any

function p, p and -ln p attain their maxima and minima, respectively, at the same point.

Applying Bayes' rule to $p(\alpha|Z_N)$ results in

$$p(\alpha|Z_N) = \frac{p(Z_N|\alpha)\ p(\alpha)}{p(Z_N)} \tag{4.1}$$

Taking the negative natural logarithm gives

$$-\ln p(\alpha|Z_N) = -\ln p(Z_N|\alpha) - \ln p(\alpha) + \ln p(Z_N) \tag{4.2}$$

Since the density function $p(Z_N)$ is not a function of α, this term in the above equation will not affect the minimization of $-\ln p(\alpha|Z_N)$ with respect to α and, therefore, need not be evaluated. The remaining terms in the equation originate from the joint (unconditional) density $p(\alpha, Z_N)$. Consequently, a derivation initiated with the density $p(\alpha, Z_N)$ would produce the same results. Nevertheless, the density functions $p(Z_N|\alpha)$ and $p(\alpha)$ must be evaluated for development of a parameter estimate.

The conditional density $p(Z_N|\alpha)$ can be defined in terms of the density function of the innovation sequence utilizing the derived density theorem in Appendix D.

The density function of the innovation sequence was shown in Chapter 2 to be Gaussian with zero mean and covariance V_i for an optimum filter with known parameters. Using this basis, the conditional density $p(Z_N|\alpha)$ is defined by the equation

$$p(Z_N|\alpha) = p(\nu_1 \ldots \nu_n)$$

$$= \frac{1}{\sqrt{(2\pi)^{Nr} \prod_{i=1}^{N} |V_i|}} \exp\left\{-\tfrac{1}{2} \sum_{i}^{N} \nu_i^T V_i^{-1} \nu_i\right\} \qquad (4.3)$$

where ν_i are the innovations and $|V_i|$ is the determinant of V_i which is equal to $H_i P_{i/i-1} H_i^T + R_i$. The density function of the parameters is also defined to be Gaussian and is distributed about the nominal α_o with covariance σ. The parameter density is

$$p(\alpha) = \frac{1}{\sqrt{(2\pi)^p |\sigma|}} \exp\left\{-\tfrac{1}{2}(\alpha-\alpha_o)^T \sigma^{-1} (\alpha-\alpha_o)\right\} \qquad (4.4)$$

where $|\sigma|$ is the determinant of σ.

The variational terms of the negative logarithm of $p(\alpha|Z_N)$, denoted $J(\alpha)$, can now be determined with the above equations. Other terms in $-\ln p(\alpha|Z_N)$ are constant and, therefore, will not affect the determination of the maximum a posteriori estimate. The quantity $J(\alpha)$ is given by the equation

$$J(\alpha) = \sum_{i}^{N} \left\{ \nu_i^T V_i^{-1} \nu_i + \ln|V_i| \right\} + (\alpha - \alpha_o)^T \sigma^{-1} (\alpha - \alpha_o) \qquad (4.5)$$

The derivative of $J(\alpha)$ with respect to the parameters α is equivalent to the derivative of $-\ln p(\alpha|Z_N)$ and is

$$\frac{\partial J(\alpha)}{\partial \alpha} = \sum_{i=1}^{N} \left\{ 2 \frac{\partial \nu_i^T}{\partial \alpha} V_i^{-1} \nu_i - \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha} V_i^{-1} \nu_i \right.$$

$$\left. + \operatorname{tr}\left[V_i^{-1} \frac{\partial V_i}{\partial \alpha} \right] \right\} + 2\sigma^{-1}(\alpha - \alpha_o) \qquad (4.6)$$

where tr is the matrix trace. Note that the partial derivatives are based on the relationships that, for any general vector a and matrix A which are functions of the parameter α,

$$\frac{\partial a^T A^{-1} a}{\partial \alpha} = 2 \frac{\partial a^T}{\partial \alpha} A^{-1} a + a^T \frac{\partial A^{-1}}{\partial \alpha} a \qquad (4.7)$$

where

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1} \qquad (4.8)$$

and

$$\frac{\partial \ln|A|}{\partial \alpha} = \frac{\partial \ln|A|}{\partial |A|} \frac{\partial |A|}{\partial \alpha}$$

$$= \frac{1}{A} \frac{\partial |A|}{\partial \alpha}$$

$$= \mathrm{tr}\ (A^{-1} \frac{\partial A}{\partial \alpha}) \qquad (4.9)$$

For a minimum value of $\partial J/\partial \alpha$ and, thus, the maximum a posteriori estimate of the parameters, this equation is set equal to zero

$$\sum_{i=1}^{N}\left\{2\,\frac{\partial\nu_i^T}{\partial\alpha}\,V_i^{-1}\nu_i - \nu_i^T V_i^{-1}\,\frac{\partial V_i}{\partial\alpha}\,V_i^{-1}\nu_i + \mathrm{tr}\left[V_i^{-1}\,\frac{\partial V_i}{\partial\alpha}\right]\right\}$$

$$+\ 2\sigma^{-1}(\alpha-\alpha_o) = 0 \qquad (4.10)$$

The solution of this matrix equation will yield improved estimates of the parameters. However, the equation does not allow a direct determination of the parameters so it is necessary to consider an iterative technique. The iterative solution for the parameter estimate is presented subsequently.

4.1.2 Expected Value of the Variational Terms of $-\ln\, p(\alpha|Z_N)$

The variational terms of $-\ln\, p(\alpha|Z_N)$ are defined by $J(\alpha)$. The expected value of $J(\alpha)$ is written

$$E[J(\alpha)] = E\left[\sum_{i}^{N} \nu_i^T V_i^{-1}\nu_i - \ln|V_i| + (\alpha-\alpha_o)^T\sigma^{-1}(\alpha-\alpha_o)\right]$$

$$= E\left[\sum_{i}^{N}\mathrm{tr}\ V_i^{-1}\nu_i\nu_i^T\right] + E\left[\sum_{i}^{N}\ln|V_i|\right]$$

$$+\ E\left[\mathrm{tr}\left\{\sigma^{-1}(\alpha-\alpha_o)(\alpha-\alpha_o)^T\right\}\right] \qquad (4.11)$$

This equation becomes

$$E[J(\alpha)] = \sum_{i}^{N} tr(I_r) + \sum_{i}^{N} \ln|V_i| + tr(I_p)$$

$$= Nr + p + \sum_{i}^{N} \ln|V_i| \tag{4.12}$$

where I_r is an identify matrix of order r which is the dimension of the innovation vector, and similarly I_p is an identity matrix of order p which is the dimension of the parameter vector. This equation provides an assessment of the expected value of $J(\alpha)$ and might be compared to the actual value of $J(\alpha)$ as a check of the estimation process.

4.2 PARAMETER SOLUTION

The derivation of the previous section lead to an equation, the solution of which yields an improved estimate of the parameters. However, this equation does not have a general closed form solution, so it is necessary to consider an iterative solution. In this section, an iterative technique is developed for the parameter solution that is based on a method known as the "method of scoring"

in the statistical literature (see Rao 1965), the modified Newton-Raphson method in the control literature, and also as the Gauss-Newton method in other contexts. The derivatives that are input to this iterative solution are also developed in this section.

4.2.1 Method of Solution

One means of obtaining a solution to the parameter equation is to employ the Newton-Raphson iteration. The philosophy of the Newton-Raphson procedure is to approximate $J(\alpha)$ as a quadratic in α by using a Taylor series expansion to second order so that

$$J(\alpha) \cong J(\alpha^*) + \left[\frac{\partial J(\alpha^*)}{\partial \alpha^*}\right]^T (\alpha-\alpha^*)$$

$$+ \tfrac{1}{2}(\alpha-\alpha^*)^T \left[\frac{\partial^2 J(\alpha^*)}{\partial \alpha^{*2}}\right](\alpha-\alpha^*) \qquad (4.13)$$

where α is the maximum a posteriori estimate and α^* is the current value. The derivative is then approximately a linear function of

$$\frac{\partial J(\alpha)}{\partial \alpha} = \frac{\partial J(\alpha^*)}{\partial \alpha^*} + \frac{\partial^2 J(\alpha^*)}{\partial \alpha^*} (\alpha-\alpha^*) \qquad (4.14)$$

For the maximum a posteriori estimate, the term $\partial J/\partial\alpha$ is set to zero which provides the iterative solution

$$\alpha = \alpha^* - \left[\frac{\partial^2 J(\alpha^*)}{\partial\alpha^{*2}}\right]^{-1}\left[\frac{\partial J(\alpha^*)}{\partial\alpha^*}\right] \tag{4.15}$$

The determined value for α is then a first-order correction to the estimate α^*. To converge to a solution, this equation is applied repeatedly until the corrections become negligibly small. (That is, α^* is used to calculate α. This α becomes α^* for the next iteration, and so on.) Unfortunately, the determination of an analytical expression for the second partial derivative of $J(\alpha)$ in the solution equation is extremely complicated and requires substantial computational time. To alleviate this difficulty, an approximation is made which simplifies the computation while maintaining accuracy over large samples.

The equation for $J(\alpha)$ originates from the variational terms of $-\ln p(Z_N|\alpha)$ and $-\ln p(\alpha)$. The value of the second partial derivative of $-\ln p(Z_N|\alpha)$ is approximated by the expected value of the product of the first partial derivatives of $-\ln p(Z_N|\alpha)$ and is known as the "method of scoring" in Rao (1965).

$$-\frac{\partial^2 \ln p(Z_N|\alpha)}{\partial\alpha^2} \cong E\left[\left\{\frac{\partial \ln p(Z_N|\alpha)}{\partial\alpha}\right\}\left\{\frac{\partial \ln p(Z_N|\alpha)}{\partial\alpha}\right\}^T\right] \qquad (4.16)$$

Rao has shown that the error committed by using this approximation is of order 1/N for large samples.

The approximation is motivated by development of derivatives of the probability density function. A basic property of the density function is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(Z_N|\alpha)\ dZ_N = 1 \qquad (4.17)$$

Performing the differentiation with respect to a parameter α_k and using the fact that p is equal to exp(ln p) gives

$$\frac{\partial}{\partial\alpha_k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(Z_N|\alpha)\ dZ_N = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\alpha_k}\ln p(Z_N|\alpha)\right] p(Z_N|\alpha)\ dZ_N$$

$$= E\left[\frac{\partial}{\partial\alpha_k}\ln p(Z_N|\alpha)\right]$$

$$= 0 \qquad (4.18)$$

Continuing the differentiation gives

$$\frac{\partial^2}{\partial\alpha_k\partial\alpha_\ell}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(Z_N|\alpha)\ dZ_N = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\alpha_k\partial\alpha_\ell}\ \ln\ p(Z_N|\alpha)\right]p(Z_N|\alpha)\ dZ_N$$

$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\alpha_k}\ \ln\ p(Z_N|\alpha)\right]\left[\frac{\partial}{\partial\alpha_k}\ \ln\ p(Z_N|\alpha)\right]p(Z_N|\alpha)\ dZ_N$$

$$= E\left[\frac{\partial^2}{\partial\alpha_k\partial\alpha_\ell}\ \ln\ p(Z_N|\alpha)\right] + E\left[\left\{\frac{\partial}{\partial\alpha_k}\ \ln\ p(Z_N|\alpha)\right\}\left\{\frac{\partial}{\partial\alpha_\ell}\ \ln\ p(Z_N|\alpha)\right\}\right]$$

$$= 0 \tag{4.19}$$

From this equation, the developed relationship is

$$-E\left[\frac{\partial^2}{\partial\alpha_k\partial\alpha_\ell}\ \ln\ p(Z_N|\alpha)\right] = E\left[\left\{\frac{\partial}{\partial\alpha_k}\ \ln\ p(Z_N|\alpha)\right\}\left\{\frac{\partial}{\partial\alpha_\ell}\ \ln\ p(Z_N|\alpha)\right\}\right] \tag{4.20}$$

The second derivative of $J(\alpha)$ utilizing this result is, therefore,

$$\frac{\partial^2 J(\alpha)}{\partial \alpha^2} \cong F(\alpha,\alpha) + 2\sigma_k^{-1}\delta_{k\ell} \tag{4.21}$$

where

$$F(\alpha,\alpha) = E\left[\left(\frac{\partial L(\alpha)}{\partial \alpha}\right)\left(\frac{\partial L(\alpha)}{\partial \alpha}\right)^T\right] \tag{4.22}$$

with

$$\frac{\partial L(\alpha)}{\partial \alpha} = \sum_i^N \left\{ \frac{2\partial \nu_i^T}{\partial \alpha} V_i^{-1}\nu_i - \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha} V_i^{-1}\nu_i + \mathrm{tr}\left(V_i^{-1}\frac{\partial V_i}{\partial \alpha}\right)\right\} \tag{4.23}$$

and where $F(\alpha,\alpha)$ is the Fisher information matrix and $\delta_{k\ell}$ is the Kronecker delta function.

To implement the iteration, it is necessary to evaluate the information matrix. Using the above equations, realizing the independence of successive innovations for a linear filter with the true parameters and recognizing that the first and third moments of a Gaussian variable are zero, allows the expression

$$F(\alpha_k, \alpha_\ell) = E\left[\left(\frac{\partial L(\alpha)}{\partial \alpha_k}\right)\left(\frac{\partial L(\alpha)}{\partial \alpha_\ell}\right)^T\right]$$

$$= \sum_i^N \Bigg\{ E\Bigg[4\, \frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V^{-1} \frac{\partial \nu_i}{\partial \alpha_\ell}$$

$$+ \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i$$

$$- \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \left[tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right)\right]$$

$$- \left[tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right)\right] \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i$$

$$+ tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right) tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right)\Bigg]\Bigg\}$$

$$= \sum_i^N \Bigg\{ E\left[4\, \frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \frac{\partial \nu_i}{\partial \alpha_\ell}\right]$$

$$+ E\left[\nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i \right]$$

$$- tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right) tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right)\Bigg\} \qquad (4.24)$$

The second term of the information matrix is evaluated in Appendix E. Utilizing the sample data, the final relationship is then

$$F(\alpha_k,\alpha_\ell) = \sum_{i}^{N} 4 \frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \frac{\partial \nu_i}{\partial \alpha_\ell} + 2\mathrm{tr}\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right) \tag{4.25}$$

Equation (4.25) together with Equations (4.21) and (4.15) thus provide the parameter estimate.

4.2.2 Derivatives of Parameter Solution

The parameter solution requires the input of two derivatives, the derivative of the innovation with respect to the parameters and the derivative of the innovation variance with respect to the parameters. The innovation is given by the equation

$$\nu_i = z_i - H_i\overline{x}_i \quad . \tag{2.42}$$

and the derivative is

$$\frac{\partial \nu_i}{\partial \alpha} = - \frac{\partial H_i}{\partial \alpha} \overline{x}_i - H_i \frac{\partial \overline{x}_i}{\partial \alpha} \tag{4.26}$$

The derivative of H_i is readily determined for any defined problem. The derivative of the expected value of x_i can be developed from the equation

$$\overline{x}_i = \phi_{i/i-1}\hat{x}_{i-1} + \psi_{i/i-1}u_{i-1}$$

$$= \phi_{i/i-1}\left\{\overline{x}_{i-1} + K_{i-1}\nu_{i-1}\right\} + \psi_{i/i-1}u_{i-1} \tag{4.27}$$

which gives the result

$$\frac{\partial \overline{x}_i}{\partial \alpha} = \frac{\partial \phi_{i/i-1}}{\partial \alpha}\,\hat{x}_{i-1} + \frac{\partial \psi_{i/i-1}}{\partial \alpha}\,u_{i-1}$$

$$+ \phi_{i/i-1}\left\{\frac{\partial \overline{x}_{i-1}}{\partial \alpha} + \frac{\partial K_{i-1}}{\partial \alpha}\,\nu_{i-1} + K_{i-1}\,\frac{\partial \nu_{i-1}}{\partial \alpha}\right\} \tag{4.28}$$

This equation demonstrates a recursive behavior in the development. To complete and evaluate the equation, the derivative of the gain must be defined. This derivative is determined from the equation

$$K_{i-1} = P_{i-1/i-2}H_{i-1}^{T}(H_{i-1}P_{i-1/i-2}H_{i-1}^{T} + R_{i-1})^{-1}$$

$$= P_{i-1/i-2}H_{i-1}^{T}V_{i-1}^{-1} \qquad (4.29)$$

and is given by

$$\frac{\partial K_{i-1}}{\partial \alpha} = \frac{\partial P_{i-1/i-2}}{\partial \alpha} H_{i-1}^{T}V_{i-1}^{-1} + P_{i-1/i-2} \frac{\partial H_{i-1}^{T}}{\partial \alpha} V_{i-1}^{-1}$$

$$- P_{i-1/i-2}H_{i-1}^{T}V_{i-1}^{-1} \frac{\partial V_{i-1}}{\partial \alpha} V_{i-1}^{-1}$$

$$= \frac{\partial P_{i-1/i-2}}{\partial \alpha} H_{i-1}^{T}V_{i-1}^{-1} + P_{i-1/i-2} \frac{\partial H_{i-1}^{T}}{\partial \alpha} V_{i-1}^{-1}$$

$$- K_{i-1} \frac{\partial V_{i-1}}{\partial \alpha} V_{i-1}^{-1} \qquad (4.30)$$

The derivatives of the conditional state error covariance and the innovation variance must now also be determined.

The derivative of the conditional state error covariance is developed from Equation (2.32) to yield

$$\frac{\partial P_{i-1/i-2}}{\partial \alpha} = \frac{\partial \phi_{i-2/i-2}}{\partial \alpha} P_{i-2}\phi^T_{i-1/i-2} + \phi_{i-1/i-2} \frac{\partial P_{i-2}}{\partial \alpha} \phi^T_{i-1/i-2}$$

$$+\phi_{i-1/i-2}P_{i-2}\left(\frac{\partial \phi_{i-1/i-2}}{\partial \alpha}\right)^T + \frac{\Gamma_{i-1/i-2}}{\partial \alpha} Q_{i-2}\Gamma^T_{i-1/i-2}$$

$$+ \Gamma_{i-1/i-2}Q_{i-2}\left(\frac{\partial \Gamma_{i-1/i-2}}{\partial \alpha}\right)^T \qquad (4.31)$$

The derivative of the state error covariance is now required. Using Equation (2.26), this derivative is

$$\frac{\partial P_{i-2}}{\partial \alpha} = \frac{-\partial K_{i-2}}{\partial \alpha} H_{i-2}P_{i-2/i-3} - K_{i-2} \frac{\partial H_{i-2}}{\partial \alpha} P_{i-2/i-3}$$

$$+ (I - K_{i-2}H_{i-2}) \frac{\partial P_{i-2/i-3}}{\partial \alpha} \qquad (4.32)$$

(If the parameter occurs only in the input matrix ψ, this derivative, the conditional error covariance derivative, and the gain derivative are zero.) These expressions produce a recursive relationship for the derivative of the innovation. The relationship requires derivatives from the penultimate time. Computationally, however, the numerical

procedure can be developed so that only derivatives from the preceding time need to be saved.

The other derivative needed in the parameter solution is the innovation variance derivative. This derivative is developed from the equation

$$V_i = H_i P_{i/i-1} H_i^T + R_i \tag{2.44}$$

The derivative is

$$\frac{\partial V_i}{\partial \alpha} = \frac{\partial H_i}{\partial \alpha} P_{i/i-1} H_i^T + H_i \frac{\partial P_{i/i-1}}{\partial \alpha} H_i^T + H_i P_{i/i-1} \frac{\partial H_i^T}{\partial \alpha} \tag{4.33}$$

and is zero when the parameter occurs only in the input matrix ψ. The derivative of H_i in the equation is readily determined while the derivative of $P_{i/i-1}$ is defined by the recursive relationship of Equations (4.30), (4.31), and (4.32)

4.3 SUMMARY OF ADAPTIVE FILTER ALGORITHMS

Previous sections provided the basis for improving the parameter estimates and, therefore, allowing a more accurate state estimate. The recursive equations of the filter are summarized in this section. The initial

algorithms parallel the Kalman filter in computation and require storage of the time histories of matrices $\phi_{i/i-1}(\alpha)$, $\psi_{i/i-1}(\alpha)$, $\Gamma_{i/i-1}(\alpha)$, $H_i(\alpha)$, Q_i, and R_i and of the computed matrices at each step $\overline{x}_i$, $\hat{x}_i$, K_i, P_i and $P_{i/i-1}$. The initial filter algorithms at time are

$$\overline{x}_i = \phi_{i/i-1}\hat{x}_{i-1} + \psi_{i/i-1}u_{i-1} \tag{4.34}$$

$$P_{i/i-1} = \phi_{i/i-1}P_{i-1}\phi^T_{i/i-1} + \Gamma_{i/i-1}Q_{i-1}\Gamma^T_{i-1} \tag{4.35}$$

$$V_i = H_iP_{i/i-1}H^T_i + R_i \tag{2.44}$$

$$K_i = P_{i/i-1}H^T_iV^{-1}_i \tag{4.36}$$

$$\hat{x}_i = (I - K_iH_i)\overline{x}_i + K_iz_i \tag{2.27}$$

$$P_i = (I - K_iH_i)P_{i/i-1}(I - K_iH_i)^T + K_iR_iK^T_i \tag{2.28}$$

The initial conditions for the computation are noted here as

$$\overline{x}_{i=o} = \overline{x}_o \tag{4.37}$$

$$P_{o/-1} = P_o \tag{4.38}$$

$$V_o = 0 \tag{4.39}$$

$$K_o = 0 \tag{4.40}$$

$$R_o = 0 \tag{4.41}$$

Note also that Q_o is not zero, but is defined by the numerical problem.

To enable an estimate of the parameters, derivatives of the Kalman filter terms must also be computed at each time. The first required derivatives are those of the conditional state estimate and the conditional error covariance which are determined prior to the Kalman filter algorithms by the equations

$$\frac{\partial \overline{x}_i}{\partial \alpha} = \frac{\partial \phi_{i/i-1}}{\partial \alpha} \hat{x}_{i-1} + \frac{\partial \psi_{i/i-1}}{\partial \alpha} u_{i-1}$$

$$+ \phi_{i/i-1} \left\{ \frac{\partial \overline{x}_{i-1}}{\partial \alpha} + \frac{\partial K_{i-1}}{\partial \alpha} \nu_{i-1} + K_{i-1} \frac{\partial \nu_{i-1}}{\partial \alpha} \right\} \tag{4.28}$$

$$\frac{\partial P_{i/i-1}}{\partial \alpha} = \frac{\partial \phi_{i/i-1}}{\partial \alpha} P_{i-1}\phi^T_{i/i-1} + \phi_{i/i-1} \frac{\partial P_{i-1}}{\partial \alpha} \phi^T_{i/i-1}$$

$$+ \phi_{i/i-1}P_{i-1} \frac{\partial \phi^T_{i-1}}{\partial \alpha} + \frac{\partial \Gamma_{i/i-1}}{\partial \alpha} Q_{i/1}\Gamma^T_{i/i-1}$$

$$+ \Gamma_{i/i-1}Q_{i-1} \frac{\partial \Gamma^T_{i/i-1}}{\partial \alpha} \tag{4.42}$$

The initial conditions for these equations are

$$\frac{\partial \overline{x}_o}{\partial \alpha} = 0 \tag{4.43}$$

$$\frac{\partial K_o}{\partial \alpha} = 0 \tag{4.44}$$

$$\frac{\partial \nu_o}{\partial \alpha} = 0 \tag{4.45}$$

$$\frac{\partial P_o}{\partial \alpha} = 0 \tag{4.46}$$

The next derivatives are computed after the Kalman filter algorithms. These derivatives, in the order of their computational sequence, are

$$\frac{\partial \nu_i}{\partial \alpha} = - \frac{\partial H_i}{\partial \alpha} \overline{x}_i - H_i \frac{\partial \overline{x}_i}{\partial \alpha} \quad (4.26)$$

$$\frac{\partial V_i}{\partial \alpha} = \frac{\partial H_i}{\partial \alpha} P_{i/i-1} H_i^T + H_i \frac{\partial P_{i/i-1}}{\partial \alpha} H_i^T + H_i P_{i/i-1} \frac{\partial H_i^T}{\partial \alpha} \quad (4.33)$$

$$\frac{\partial K_i}{\partial \alpha} = \frac{\partial P_{i/i-1}}{\partial \alpha} H_i^T V_i^{-1} + P_{i/i-1} \frac{\partial H_i^T}{\partial \alpha} V_i^{-1} - K_i \frac{\partial V_i}{\partial \alpha} V_i^{-1} \quad (4.47)$$

$$\frac{\partial P_i}{\partial \alpha} = - \frac{\partial K_i}{\partial \alpha} H_i P_{i/i-1} - K_i \frac{\partial H_i}{\partial \alpha} P_{i/i-1} + (I - K_i H_i) \frac{\partial P_{i/i-1}}{\partial \alpha}$$

(4.48)

The accumulated score S_i and the information $F_i(\alpha_k, \alpha_\ell)$ are also determined at each time point by the relationships

$$S_i = 2 \frac{\partial \nu_i^T}{\partial \alpha} V_i^{-1} \nu_i - \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha} V_i^{-1} \nu_i + \mathrm{tr}\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha}\right) + S_{i-1}$$

(4.49)

$$F_i(\alpha_k,\alpha_\ell) = 4\,\frac{\partial \nu_i^T}{\partial\alpha}\,V_i^{-1}\,\frac{\partial \nu_i}{\partial\alpha} + 2\,\mathrm{tr}\left(V_i^{-1}\,\frac{\partial V_i}{\partial\alpha}\,V_i^{-1}\,\frac{\partial V_i}{\partial\alpha}\right)$$

$$+ F_{i-1}(\alpha_k,\alpha_\ell) \qquad (4.50)$$

These quantities are initially zero and are determined only every N samples so that, when i is equal to N, an estimate of the parameters is made and the quantities are reset to zero. The parameter estimate α is made by the relationship

$$\alpha = \alpha^* - \left[\frac{\partial^2 J(\alpha^*)}{\partial\alpha^{*2}}\right]^{-1}\left[\frac{\partial J(\alpha^*)}{\partial\alpha^*}\right] \qquad (4.15)$$

where

$$\frac{\partial^2 J(\alpha^*)}{\partial\alpha^{*2}} = F_N(\alpha^*,\alpha^*) + 2\sigma_k^{-1}\delta_{k\ell} \qquad (4.51)$$

$$\frac{\partial J(\alpha^*)}{\partial\alpha^*} = S_N + 2\sigma^{-1}(\alpha^*-\alpha_o) \qquad (4.52)$$

and where α^* is the current parameter estimate. After the estimate, the last N measurements are reprocessed with the algorithms to establish a new parameter estimate. This

process is repeated until the corrections to the parameter estimate become negligibly small. At this point, the state estimates produced by the filter are accepted. Also, after each cycle of N measurements, derivatives are reinitialized so that

$$\frac{\partial \overline{x}_i}{\partial \alpha} = 0 \tag{4.53}$$

$$\frac{\partial K_i}{\partial \alpha} = 0 \tag{4.54}$$

$$\frac{\partial \nu_i}{\partial \alpha} = 0 \tag{4.55}$$

$$\frac{\partial P_i}{\partial \alpha} = 0 \tag{4.56}$$

This procedure is termed batch processing. The data from the previous batch is considered as a priori information for the current batch and so on until the estimates are accepted.

4.4 CONVERGENCE OF ESTIMATE

Consider now the convergence of the parameter

estimate. The estimate is defined with the derivative of $J(\alpha)$ set to zero or

$$\frac{\partial J(\alpha)}{\partial \alpha} = 0 \tag{4.57}$$

This derivative can be redefined as

$$\frac{\partial J(\alpha)}{\partial \alpha} = {}_{\alpha}J(Z_N,\alpha) = \sum_{i}^{N} {}_{\alpha}J_i(z_i,\alpha) \tag{4.58}$$

Now let the expected value ${}_{\alpha}\overline{J}(z_i,\alpha|\alpha_t)$ of the individual terms ${}_{\alpha}J_i(z_i,\alpha)$, conditioned on the true parameters, be defined as

$$\begin{aligned} {}_{\alpha}\overline{J}_i(z_i,\alpha|\alpha_t) &= E\left[{}_{\alpha}J_i(z_i,\alpha|\alpha_t)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}_{\alpha}J_i(z_i,\alpha) f(z_i|\alpha_t)\, dz_i \end{aligned} \tag{4.59}$$

where α_t if the true value of the parameter, so that the average of the individual values if

$$ {}_{\alpha}\overline{J}(Z_N,\alpha|\alpha_t) = \frac{1}{N}\sum_{i}^{N} \overline{J}_i(z_i,\alpha|\alpha_t) \tag{4.60}$$

Therefore, ${}_{\alpha}J(Z_N,\alpha)$ is the sample mean from a population whose mean is ${}_{\alpha}\overline{J}(Z_N,\alpha|\alpha_t)$. By the weak law of large numbers, ${}_{\alpha}J(Z_N,\alpha)$ converges in probability to ${}_{\alpha}J(Z_N,\alpha|\alpha_t)$ which, for an arbitrary value of ϵ greater than zero, is written

$$\lim_{N\to\infty} P\left\{\left|{}_{\alpha}J(Z_N,\alpha) - {}_{\alpha}\overline{J}(Z_N,\alpha|\alpha_t)\right| \geq \epsilon\right\} = 0 \tag{4.61}$$

Since the parameter estimate is based on minimizing $J(\alpha)$, the mean value ${}_{\alpha}\overline{J}(Z_N,\alpha|\alpha_t)$ is strictly increasing over a range of α including α_t. Let the range be $\alpha_t - \delta$ to $\alpha_t + \delta$ where δ is some positive value. If α is set equal to α_t, then

$$ {}_{\alpha}\overline{J}(Z_N,\alpha_t|\alpha_t) = E\left[{}_{\alpha}J(Z_N,\alpha_t)|\alpha_t\right] = 0 \tag{4.62}$$

Consequently, when α is $\alpha_t-\delta$, the value of ${}_{\alpha}\overline{J}(Z_N,\alpha|\alpha_t)$ is given by the inequality

$$ {}_{\alpha}J(Z_N,\alpha_t-\delta|\alpha_t) < 0 \tag{4.63}$$

Similarly, when α is $\alpha_t+\delta$, the inequality of ${}_{\alpha}\bar{J}(Z_N,\alpha\,|\,\alpha_t)$ is

$$
{}_{\alpha}J(Z_N,\alpha_t+\delta\,|\,\alpha_t) > 0 \tag{4.64}
$$

Based on these results and Equation (4.60), the sample mean can be defined with probability exceeding $1 - \epsilon$. and, requiring some undetermined number of samples which may depend on δ and ϵ by the inequalities

$$
{}_{\alpha}J(Z_N,\alpha) < 0 \text{ if } \alpha = \alpha_t-\delta \tag{4.65}
$$

and

$$
{}_{\alpha}J(Z_N,\alpha) > 0 \text{ if } \alpha = \alpha_t+\delta \tag{4.66}
$$

Therefore, for some α in $\alpha_t\pm\delta$, the probability that ${}_{\alpha}J(Z_N,\alpha)$ equals zero is

$$
P\left\{{}_{\alpha}J(Z_N,\alpha) = 0\,|\,\alpha_t\right\} = 1 - \epsilon \tag{4.67}
$$

Thus, there exists a solution for α which converges in probability to α_t as the sample size increases.

CHAPTER 5

NUMERICAL RESULTS

This chapter applies the filter estimation techniques of the previous chapters to a numerical example. The example consists of estimating the attitude rates of a symmetric spinning body, assuming an inexact knowledge of the spin rate. Results are derived for both the modified Kalman filter and the adaptive filter presented in the previous chapters. Numerical results are also generated for the Kalman and extended Kalman filters to provide a comparative assessment. The numerical example and the four filter formulations are delineated in the first sections of the chapter. The last sections provide the numerical results and comparisons.

5.1 NUMERICAL EXAMPLE

The angular motion of a moment-free spinning body is described by a set of equations known as Euler's equations. These rotational equations of motion can be derived by applying the principle of conservation of angular momentum. That is, the time derivative in inertial space of the angular momentum vector determined about the system mass center is equal to zero. For a symmetric body

spinning at a constant rate about the principal axis of maximum moment of inertia, designated here as axis 3, the moment-free equations of motion are:

$$\dot{\omega}_1 + \lambda\omega_2\omega_3 = 0 \tag{5.1}$$

$$\dot{\omega}_2 - \lambda\omega_1\omega_3 = 0 \tag{5.2}$$

where

$$\lambda = \frac{I_3}{I_1} - 1 \tag{5.3}$$

and

ω_1 - angular rate about body axis 1 of an orthogonal set of body-fixed axes

ω_2 - angular rate about body axis 2 of an orthogonal set of body-fixed axes

ω_3 - angular spin rate of body (assumed to be constant)

I_1 - principal moment of inertia about axis 1 (also moment of inertia about body axis 2 for symmetric body)

I_3 - maximum moment of inertia for principal axis 3

λ - moment-of-inertia factor

These equations can be defined by a discrete representation utilizing Appendix A. Assuming a short sample period relative to the natural frequency of motion and allowing disturbances in the form of additive white noise, the equations of motion become

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & -\omega_3\tau \\ \omega_3\tau & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_i \tag{5.4}$$

where

$$\tau = \lambda T \tag{5.5}$$

and where T is the sample period and w is the disturbance noise.

Spin-stabilized spacecraft typically spin at 60 rpm or 6.3 rad/sec and, because of volume and packaging constraints, have moment-of-inertia factors λ of approximately 0.1 [e.g., Williamson (1974) reports a spin rate of 60 rpm for the FLTSATCOM satellite and a moment-of-inertia factor of about 0.1]. Therefore, let the nominal value for ω_3 be 6.3 rad/sec and let λ be 0.1. Also, let the sample

period be 0.1 sec so that τ is 0.01 sec and $\omega_3\tau$ is nominally 0.063. Let the variance of the spin rate be 0.3 $(\text{rad/sec})^2$ and set the true value of the spin rate at 7.4 rad/sec (i.e., a value roughly equivalent to twice the standard deviation). Consequently, the true system dynamics are given by

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & -0.074 \\ 0.074 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_i \tag{5.6}$$

and the assumed system dynamics are

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & -0.063 \\ 0.063 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{5.7}$$

Let the statistics on the initial conditions be defined as

$$\begin{bmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix}_0 = \begin{bmatrix} 0.63 \\ 0 \end{bmatrix} \text{rad/sec} \tag{5.8}$$

$$[P]_0 = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix} (\text{rad/sec})^2 \qquad (5.9)$$

and let the actual initial conditions be set at approximately 3σ values in order to observe the transient behavior of the system, or

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_0 = \begin{bmatrix} 0.66 \\ 0.03 \end{bmatrix} \text{rad/sec} \qquad (5.10)$$

Assume the noise covariance of the system to be time invariant and given by

$$Q_i = 0.00001 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{rad/sec})^2 \qquad (5.11)$$

Let the measurements be related to the angular rates with additive white noise by the equation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i \qquad (5.12)$$

and assume the covariance of the measurement noise to be time invariant and given by

$$R_i = 0.0005 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{rad/sec})^2 \qquad (5.13)$$

Thus, these relationships now define a numerical example that can illustrate the filter estimation techniques of the previous chapters. In addition, the assumed values are consistent with physical constraints of spinning bodies identified in Appendix F.

5.2 KALMAN FILTER FORMULATION

The state to be estimated by the Kalman filter is designated x_i and given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_i = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_i \qquad (5.14)$$

The predicted value of the state based on the nominal values of the transition matrix is

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}_i = \phi_{i/i-1}\hat{x}_{i-1}$$

$$= \begin{bmatrix} 1 & -0.063 \\ 0.063 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_{i-1} \tag{5.15}$$

where the error covariance is given by

$$P_{i/i-1} = \phi_{i/i-1} P_{i-1} \phi^T_{i/i-1} + Q_{i-1}$$

$$= \begin{bmatrix} 1 & -0.063 \\ 0.063 & 1 \end{bmatrix} P_{i-1} \begin{bmatrix} 1 & -0.063 \\ 0.063 & 1 \end{bmatrix}^T$$

$$+ \begin{bmatrix} 0.00001 & 0 \\ 0 & 0.00001 \end{bmatrix} \tag{5.16}$$

The iritial values input to filter predictions are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}_0 = \begin{bmatrix} 0.31 \\ 0 \end{bmatrix} \tag{5.17}$$

$$P_0 = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix} \tag{5.18}$$

The gain equation for the filter estimate is given by

$$\begin{aligned} K_i &= P_{i/i-1}\left[P_{i/i-1} + R_i\right]^{-1} \\ &= P_{i/i-1}\left[P_{i/i-1} + \begin{pmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{pmatrix}\right]^{-1} \end{aligned} \tag{5.19}$$

The filter estimate is

$$\hat{x}_i = \left[I - K_i\right]\bar{x}_i + K_i z_i \tag{5.20}$$

where the measurements z_i are derived from

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_i = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_i + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i \tag{5.21}$$

and the state x_i is determined from the true system dynamics

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_i = \begin{bmatrix} 1 & -0.074 \\ 0.074 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{i-1} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_{i-1} \tag{5.22}$$

with the true initial conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_0 = \begin{bmatrix} 0.66 \\ 0.03 \end{bmatrix} \tag{5.23}$$

Finally, the covariance of the filter error is

$$P_i = \left[I-K_i\right] P_{i/i-1} \left[I-K_i\right]^T + K_i R_i K_i^T$$

$$= \left[I-K_i\right] P_{i/i-1} \left[I-K_i\right]^T + K_i \begin{bmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{bmatrix} K_i^T \tag{5.24}$$

These quantities determined at time i become the inputs to the filter equations for the estimate of the state at time i+1. Thus, the equations form a recursive set of expressions that process the measurements as they become available to the filter.

5.3 MODIFIED KALMAN FILTER FORMULATION

The modified Kalman filter operates in a similar manner to the Kalman filter with the exception that the gain and error covariance matrices are altered to reflect the uncertainty in the system parameters. The modification in these matrices is accomplished by utilizing the parameter covariance and the derivative of the system matrices with respect to the parameter. For this example, the uncertain parameter α is the body spin rate ω_3. The derivative of the transition matrix with respect to the spin rate is

$$_{\alpha_1}\phi_{i/i-1} = \begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix} \tag{5.25}$$

and the spin rate covariance σ is 0.3 $(\text{rad/sec})^2$. The terms of the gain matrix become

$$_{\alpha_1}r_i = \sigma_{\alpha_1}\phi_{i/i-1}\left[P_{i-1} + \hat{x}_{i-1}\hat{x}^T_{i-1}\right]_{\alpha_1}\left[\phi_{i/i-1}\right]^T$$

$$= 0.3\begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix}\left[P_{i-1} + \hat{x}_{i-1}\hat{x}^T_{i-1}\right]\begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix}^T \tag{5.26}$$

$$S_i = 0 \tag{5.27}$$

$$J_i = P_{i/i-1} + \sum_{k=1}^{p=1} \alpha_k r_i$$

$$= P_{i/i-1} + 0.3 \begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix} \left[P_{i-1} + \hat{x}_{i-1} \hat{x}_{i-1}^T \right] \begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix}^T \tag{5.28}$$

and the gain matrix is

$$G_i = J_i (J_i + R_i + S_i)^{-1}$$

$$= J_i \left[J_i + \begin{pmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{pmatrix} \right]^{-1} \tag{5.29}$$

The filter estimate is

$$\hat{x}_i = (I - G_i)\overline{x}_i + G_i z_i \tag{5.30}$$

and the error covariance of the filter estimate is given by

$$P_i = (I-G_i)P_{i/i-1}(I-G_i)^T + G_iR_iG_i^T + T_i$$

$$= (I-G_i)P_{i/i-1}(I-G_i)^T + G_i\begin{bmatrix}0.0005 & 0\\ 0 & 0.0005\end{bmatrix}G_i^T + T_i \quad (5.31)$$

where

$$T = \sum_{k=1}^{p=1} \alpha_k t_i$$

$$= \sigma_{\alpha_1}(\phi_{i/i-1}-G_i\phi_{i/i-1})(P_{i-1}+\hat{x}_{i-1}\hat{x}_{i-1}^T)_{\alpha_1}(\phi_{i/i-1}-G_i\phi_{i/i-1})^T$$

$$= 0.3\ (I-G_i)\begin{bmatrix}0 & -0.01\\ 0.01 & 0\end{bmatrix}(P_{i-1} + \hat{x}_{i-1}\hat{x}_{i-1}^T)$$

$$\begin{bmatrix}0 & -0.01\\ 0.01 & 0\end{bmatrix}^T(I-G_i)^T \quad (5.32)$$

Note that, in this filter, the state estimate uses one less equation and the error covariance matrices require five less equations than the extended Kalman filter.

5.4 EXTENDED KALMAN FILTER FORMULATION

In the extended Kalman filter, the uncertain parameter in the system dynamics is augmented to the state vector so that

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}_i = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_i \tag{5.33}$$

The system dynamics are then

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}_i = \begin{bmatrix} x' - 0.01x_2'x_3' \\ x_2' + 0.01x_1'x_3' \\ x_3' \end{bmatrix}_{i-1} + \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix}_{i-1} \tag{5.34}$$

From the development in Chapter 2, the predicted value of the augmented state is written

$$\begin{bmatrix} \bar{x}_1' \\ \bar{x}_2' \\ \bar{x}_3' \end{bmatrix}_i = \begin{bmatrix} \hat{x}_1' - 0.01\hat{x}_2'\hat{x}_3' \\ \hat{x}_2' + 0.01\hat{x}_1'\hat{x}_3' \\ \hat{x}_3' \end{bmatrix}_{i-1} \tag{5.35}$$

and the covariance of the predicted state becomes

$$P'_{i/i-1} = \phi'_{i/i-1} P'_{i-1} \phi'^{T}_{i/i-1} + Q_{i-1}$$

$$= \begin{bmatrix} 1 & -0.01\hat{x}'_3 & -0.01\hat{x}'_2 \\ 0.01\hat{x}'_3 & 1 & 0.01\hat{x}'_1 \\ 0 & 0 & 1 \end{bmatrix} P'_{i-1}$$

$$\begin{bmatrix} 1 & -0.01\hat{x}'_3 & -0.01\hat{x}'_2 \\ 0.01\hat{x}'_3 & 1 & 0.01\hat{x}'_1 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$+ \begin{bmatrix} 0.00001 & 0 & 0 \\ 0 & 0.00001 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.36}$$

The initial estimates provided to the filter are

$$\begin{bmatrix} \hat{x}'_1 \\ \hat{x}'_2 \\ \hat{x}'_3 \end{bmatrix}_0 = \begin{bmatrix} 0.63 \\ 0 \\ 6.3 \end{bmatrix} \tag{5.37}$$

$$P_o' = \begin{bmatrix} 0.0001 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \tag{5.38}$$

The gain equation of the extended filter is

$$K_i' = P_{i/i-1}' H_i'^T (H_i' P_{i/i-1}' H_i'^T + R_i)^{-1}$$

$$= P_{i/i-1}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} P_{i/i-1}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T + \begin{bmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{bmatrix} \right\}^{-1} \tag{5.39}$$

The filter estimate using this gain is

$$\hat{x}_i' = \bar{x}_i' + K_i'(z_i - \bar{x}_i') \tag{5.40}$$

and the error covariance of the estimate is given by

$$P_i' = (I - K_i' H_i') P_{i/i-1}' (I - K_i' H_i')^T + K_i' R_i K_i'^T$$

$$= \left\{ I - K_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\} P_{i/i-1}' \left\{ I - K_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}^T + K_i' \begin{bmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{bmatrix} K_i^T \tag{5.41}$$

The measurements are given by the equation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1' x_2' x_3' \end{bmatrix}^T + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{5.42}$$

where the state is based on the true system dynamics

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & -0.074 & 0 \\ 0.074 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_i \begin{bmatrix} x_1' \\ x_2' \\ 7.4 \end{bmatrix}_i + \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix}_i \tag{5.43}$$

with the true initial conditions

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0.66 \\ 0.03 \\ 7.4 \end{bmatrix} \tag{5.44}$$

The extended Kalman filter operates with the above equations in a recursive manner estimating both the states and the uncertain parameter of the system dynamics.

5.5 ADAPTIVE FILTER FORMULATION

The adaptive filter uses the algorithms of the Kalman filter and derivatives of the filter terms to perform the parameter estimation and improve the state estimates. The filter algorithms operate with the current estimate of the body spin rate designated α^*. The initial value assumed for the spin rate is 6.3 rad/sec. The process is initiated by computing the two derivatives

$$\frac{\partial \bar{x}_i}{\partial \alpha} = \frac{\partial \phi_{i/i-1}}{\partial \alpha} \hat{x}_{i-1} + \phi_{i/i-1} \left\{ \frac{\partial \bar{x}_{i-1}}{\partial \alpha} + \frac{\partial K_{i-1}}{\partial \alpha} \nu_{i-1} + K_{i-1} \frac{\partial \nu_{i-1}}{\partial \alpha} \right\}$$

$$= \begin{bmatrix} 0 & -0.01 \\ 0.01 & 0 \end{bmatrix} \hat{x}_{i-1} + \begin{bmatrix} 1 & -0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix}$$

$$\left\{ \frac{\partial \bar{x}_{i-1}}{\partial \alpha} + \frac{\partial K_{i-1}}{\partial \alpha} \nu_{i-1} + K_{i-1} \frac{\partial \nu_{i-1}}{\partial \alpha} \right\} \tag{5.45}$$

$$\frac{\partial P_{i/i-1}}{\partial \alpha} = \frac{\partial \phi_{i/i-1}}{\partial \alpha} P_{i-1} \phi^T_{i/i-1} + \phi_{i/i-1} \frac{\partial P_{i-1}}{\partial \alpha} \phi^T_{i/i-1}$$

$$+ \phi_{i/i-1} P_{i-1} \frac{\partial \phi^T_{i/i-1}}{\partial \alpha}$$

(cont)

$$= \begin{bmatrix} 0 & -\ 0.01 \\ 0.01 & 0 \end{bmatrix} P_{i-1} \begin{bmatrix} 1 & -\ 0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix}^T$$

$$+ \begin{bmatrix} 1 & -\ 0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix} \frac{\partial P_{i-1}}{\partial \alpha} \begin{bmatrix} 1 & -\ 0.01\ * \\ 0.01\alpha^* & 1 \end{bmatrix}^T$$

$$+ \begin{bmatrix} 1 & -\ 0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix} P_{i-1} \begin{bmatrix} 0 & -\ 0.01 \\ 0.01 & 0 \end{bmatrix}^T \qquad (5.46)$$

The initial values of the derivatives in these equations are zero. In addition, these derivatives are set to zero after processing every N measurements for the parameter estimate. In this example, N is set equal to 30.

Following computation of the above equations, the Kalman filter algorithms are utilized. These algorithms are

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}_i = \phi_{i/i-1} \hat{x}_{i/i-1}$$

$$= \begin{bmatrix} 1 & -\ 0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{i-1} \qquad (5.47)$$

$$P_{i/i-1} = \phi_{i/i-1} P_{i-1} \phi^T_{i/i-1} + Q_{i-1}$$

$$= \begin{bmatrix} 1 & -0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix} P_{i-1} \begin{bmatrix} 1 & -0.01\alpha^* \\ 0.01\alpha^* & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.00001 & 0 \\ 0 & 0.00001 \end{bmatrix} \tag{5.48}$$

$$V_i = P_{i/i-1} + R_i$$

$$= P_{i/i-1} + \begin{bmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{bmatrix} \tag{5.49}$$

$$K_i = P_{i/i-1} V_i^{-1}$$

$$= P_{i/i-1} \left[P_{i/i-1} + \begin{pmatrix} 0.0005 & 0 \\ 0 & 0.0005 \end{pmatrix} \right]^{-1} \tag{5.50}$$

$$\hat{x}_i = (I - K_i)\bar{x}_i + K_i z_i \tag{5.51}$$

$$P_i = (I-K_i)P_{i/i-1}(I-K_i)^T + K_iR_iK_i^T$$

$$= (I-K_i)P_{i/i-1}(I-K_i)^T + K_i\begin{bmatrix}0.0005 & 0\\ 0 & 0.0005\end{bmatrix}K_i^T \qquad (5.52)$$

where the initial conditions and measurements are identical to those of the Kalman filter presented in a previous section.

The following derivatives are also computed at each time sample

$$\frac{\partial \nu_i}{\partial \alpha} = -\frac{\partial \bar{x}_i}{\partial \alpha} \qquad (5.53)$$

$$\frac{\partial V_i}{\partial \alpha} = \frac{\partial P_{i/i-1}}{\partial \alpha} \qquad (5.54)$$

$$\frac{\partial K_i}{\partial \alpha} = \frac{\partial P_{i/i-1}}{\partial \alpha} V_i^{-1} - K_i \frac{\partial V_i}{\partial \alpha} V_i^{-1} \qquad (5.55)$$

$$\frac{\partial P_i}{\partial \alpha} = -\frac{\partial K_i}{\partial \alpha} P_{i/i-1} + (I-K_i)\frac{\partial P_{i/i-1}}{\partial \alpha} \qquad (5.56)$$

These derivatives form the accumulated score and information determined by the equations

$$S_i = 2\,\frac{\partial \nu_i^T}{\partial \alpha}\,V_i^{-1}\nu_i + \nu_i^T V_i^{-1}\,\frac{\partial V_i}{\partial \alpha}\,V_i^{-1}\nu_i$$

$$+ \operatorname{tr}\left[V_i^{-1}\,\frac{\partial V_i}{\partial \alpha}\right] + S_{i-1} \tag{5.57}$$

$$F_i = 4\,\frac{\partial \nu_i^T}{\partial \alpha}\,V_i^{-1}\,\frac{\partial \nu_i}{\partial \alpha} + 2\operatorname{tr}\left[V_i^{-1}\,\frac{\partial V_i}{\partial \alpha}\,V_i^{-1}\,\frac{\partial V_i}{\partial \alpha}\right] + F_{i-1} \tag{5.58}$$

The parameter estimate is then determined by

$$\alpha = \alpha^* - \left\{F_N + 2\sigma^{-1}\right\}^{-1}\left\{S_N + 2\sigma^{-1}(\alpha^* - \alpha_o)\right\}$$

$$= \alpha^* - \left\{F_N + \frac{20}{3}\right\}^{-1}\left\{S_N + \frac{20}{3}\,(\alpha^* - 6.3)\right\} \tag{5.59}$$

After the parameter estimate, the last N measurements are reprocessed repeatedly until the correction to the parameter estimate is negligibly small (0.01 in this example). At this point, the state estimates are accepted and the next series of measurements is processed. To summarize, the

Kalman filter algorithms continue to estimate the two system states using the current estimate of the uncertain parameter while the additional algorithms adaptively determine the uncertain parameter from the innovations.

5.6 NUMERICAL COMPARISON

The state estimates and the errors in the estimates are graphically displayed in the following figures. Figures 1 through 4 show the estimate of state 1 compared with the true value of the state for the Kalman filter, the modified Kalman filter, the extended Kalman filter, and the adaptive filter, respectively. Both the estimate and the true value of the state have been normalized to the maximum value of the state. In Figures 5 through 8, the percentage difference between the normalized estimate and the state for each filter is presented. The Kalman filter produced a maximum error in the estimate of about 8 percent while the modified Kalman filter reduced the maximum error to about 6 percent. By estimating the uncertain parameter in the system dynamics (i.e., state 3), the extended Kalman and adaptive filters dramatically reduced the error. The maximum error of the extended Kalman filter was approximately 2 percent while the adaptive filter was about the

same but, at certain times, slightly larger. Similar filtering results for state 2 are presented in Figures 9 through 16.

Only the extended Kalman and adaptive filters provided estimates of state 3. These results are shown in Figures 17 through 20. Figures 17 and 18 present the normalized state 3 estimates for the two filters, and Figures 19 and 20 show the normalized percentage error in the estimates. The extended Kalman filter provided a better estimate of state than the adaptive filter except during the initial sample period. During the initial period, the adaptive filter produced a more accurate estimate. This result is more significant if the error in the initial estimate of state 3 is larger.

In Figures 21 through 33, the initial error in the estimate of state 3 is increased by setting the state 3 value equal to zero for the two filters. During the initial period, the adaptive filter generated more accurate estimates for both the uncertain parameter and the two system states. This is due to the fact that the adaptive filter operates as a fixed-time-lag data smoother (i.e., 30 data samples are acquired prior to establishing the first state estimate) while the extended Kalman filter operates as a

true filter (i.e., generating the first state estimate as the first data sample is obtained).

The extended Kalman filter, however, provided slightly better estimates during later periods than the adaptive filter and did not display any signs of divergence (i.e., instability) that exist for the extended Kalman filter in some problems. For example, divergence of the extended Kalman filter has been particularly acute in orbit determination problems due, often, to system modeling errors (see Wolf 1968). The divergence in the filter can occur when the error covariance matrix becomes very small. Since the filter gain is related to the error covariance, it also becomes very small and the filter estimate becomes decoupled from the observational sequence such that the estimate is not affected by a growing observational error.

The computational time for operation of the Kalman filter on the Control Data Corporation 7600 computer for the 300 measurement samples was 1.86 seconds. The times for the modified Kalman, extended Kalman, and adaptive filters were 1.90, 2.34, and 9.56 seconds, respectively. The modified Kalman filter provided more accurate state estimates in about the same time as the Kalman filter. The extended Kalman filter, however, produced more accurate

estimates than the modified Kalman filter with a slight increase in computational time. The adaptive filter did not improve the accuracy relative to the extended Kalman filter with the exception of the initial sample period and consumed a relatively larger computational time. The larger time for the adaptive filter was due to the number of iterations required to solve the nonlinear estimation equation for state 3. It is interesting to note, however, that a significant portion of computer time (i.e., about 4 additional seconds for each filter) was required just for the printing of results. Therefore, the additional computational time may not be significant compared to the total computer time.

5.7 EFFECT OF NOISE LEVEL

The effect of the level of measurement noise on the various filter estimates is illustrated in Figures 33 through 72. In Figures 33 through 52 the measurement noise covariance is reduced by a factor of 2, and in Figures 53 through 72 the covariance is increased by a factor of 2. In general, the error in all the filter estimates was reduced when the noise was decreased and was increased when the noise was increased. However, this reduction and enlargement in the filter error was more

significant in the Kalman and modified Kalman filters. The extended Kalman and adaptive filters showed little difference in the estimate error with variation in measurement noise level.

5.8 DATA ABERRATIONS

The effects of a gap in the measurement data and an erroneous data spike were examined for the modified Kalman filter and the adaptive filter. Data for samples 100 through 104 was eliminated from the filtering process. The effect on the two filter estimates is shown in Figures 73 through 82. In general, the error in the estimate generated by the modified Kalman filter was increased by the data gap beginning at the time of the first missing sample. The increase in the filter error, however, was only temporary and in this example the error magnitude became equivalent to previous results after approximately 20 samples. In contrast, the effect of the data gap on the adaptive filter was small and differed little from results generated with the full complement of data.

The data at sample 100 was multiplied by a factor of 10 to create an erroneous spike in the data with a variation equivalent to more than 100 times the standard deviation of the measurement noise. Figures 83 through 92

illustrate the effect on the estimation process for the two filters. The error at sample 100 of the modified Kalman filter estimate was significantly increased and remained greater than previous results for approximately the next 25 samples. After this sample period, the estimates generated by the modified Kalman filter did not differ from previous results.

Because of the iterative nature of the adaptive filter, the error for sample estimates prior to sample 100 was increased by the data spike at sample 100. In addition, the estimate error was abnormally higher for approximately the next 50 to 60 samples. After this period, the estimate error was equivalent to results generated without the data spike. Due to the dramatic effect of the data spike on the estimate error, practical use of the adaptive filter may require a pre-filter to limit data spikes and ameliorate the effects on the filter estimation.

The data spike might be considered as noise and, as such, be indicative of a larger noise covariance. A larger covariance could then be substituted into the filter algorithms. The effect of this substitution would be to decrease the gain term and make the filter less responsive to data spikes, and thus, provide a better estimate for

these samples. The filter, however, would also be less responsive to samples without the data spike. Since samples without data spikes reflect a lower noise covariance and because the filter is only optimum on the basis of an assumed white noise process with known covariance, the filter would not provide the best estimates for these samples. By retaining the lower noise covariance representive of the measurement noise without data spikes, better estimates can be made for these estimates while degraded estimates will be made of samples with data spikes. Also, since the effect of the data spike was shown in this example to attentuate with time, retaining the lower noise covariance may be more desirable.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\mathrm{rad/sec})^2$
System Noise Covariance - 0.00001 $(\mathrm{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\mathrm{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

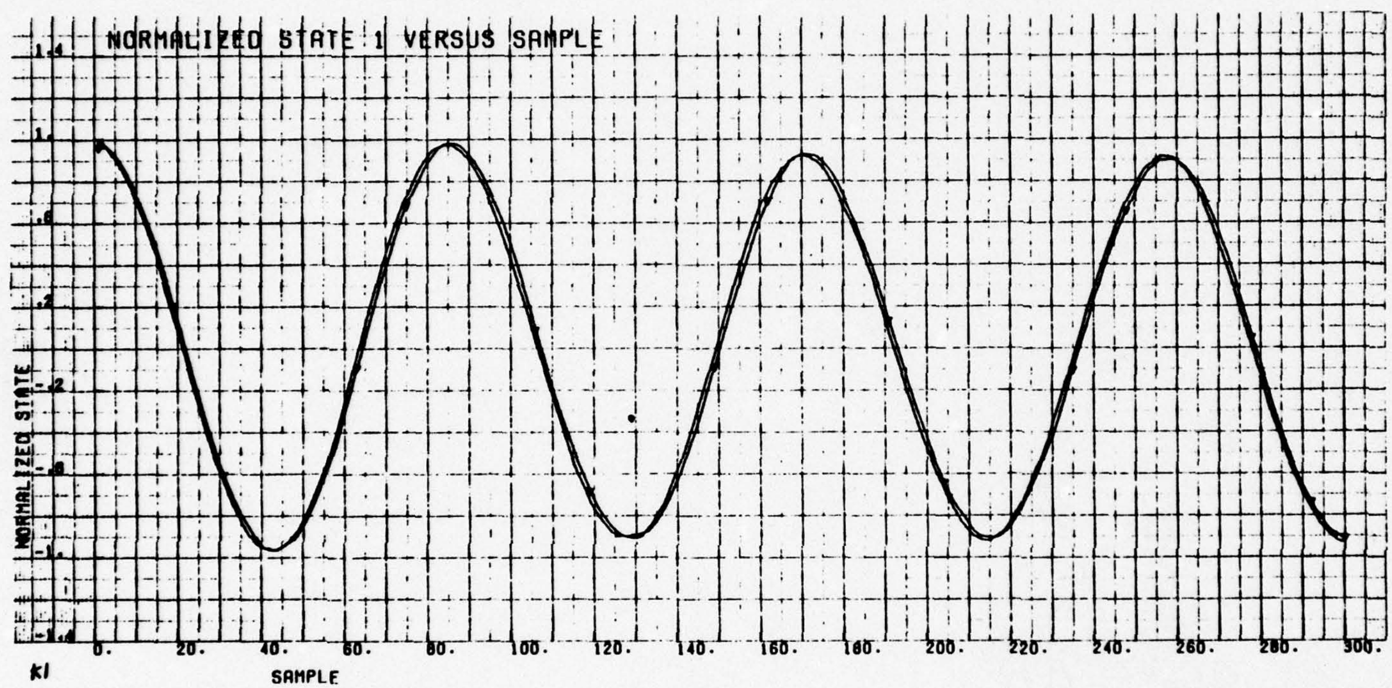

Figure 1. Kalman Filter Estimate of State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ___
Estimated State

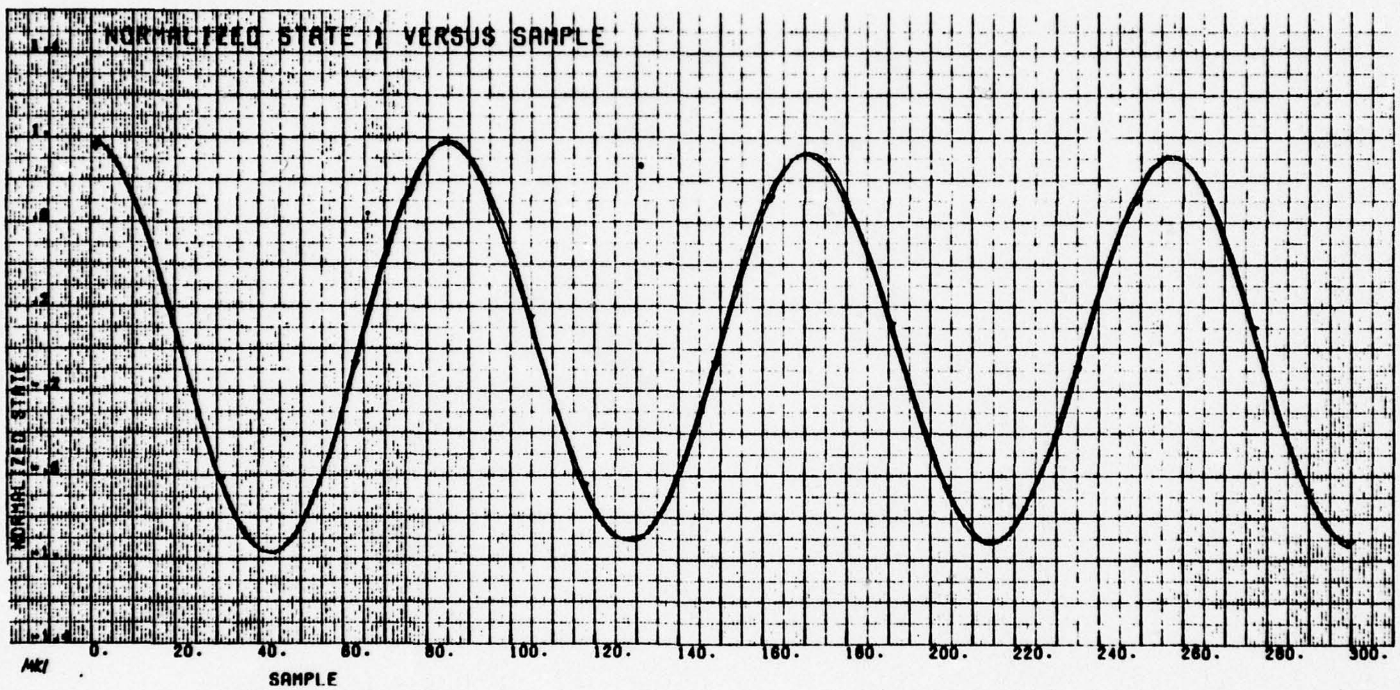

Figure 2. Modified Kalman Filter Estimate of State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ———
Estimated State

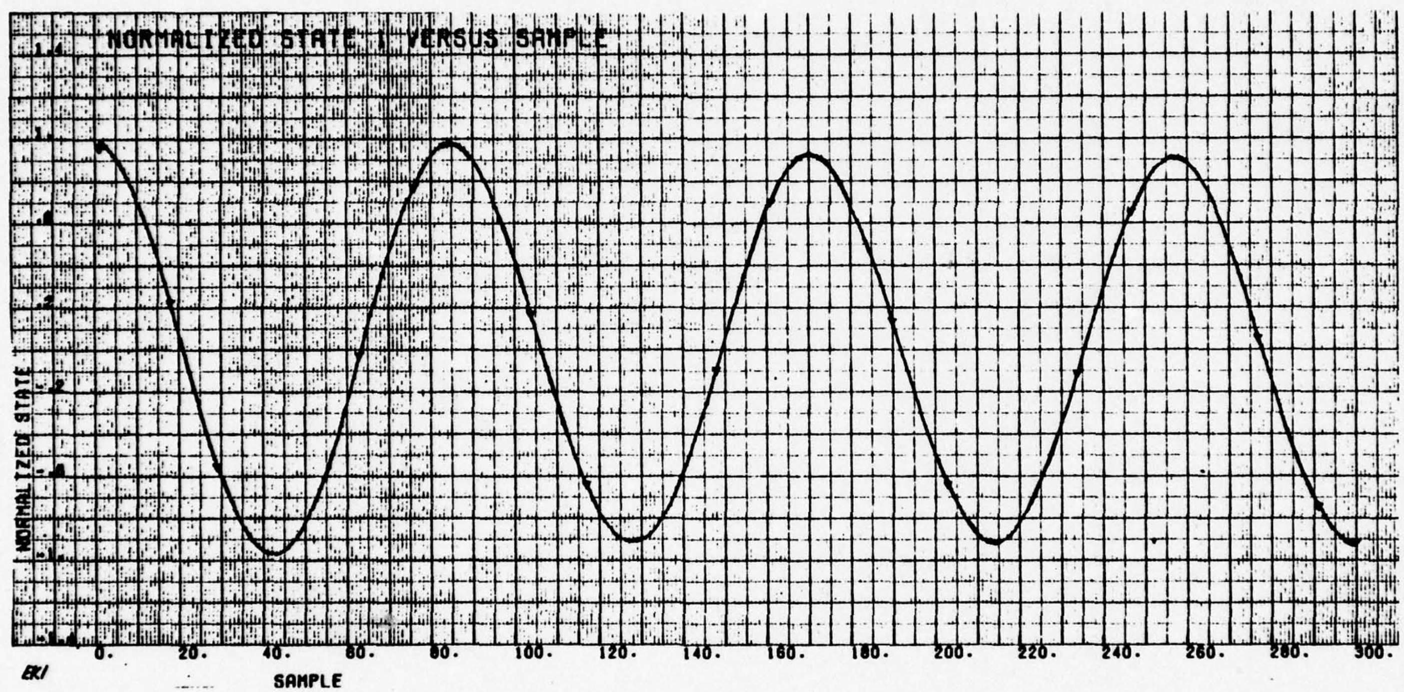

Figure 3. Extended Kalman Filter Estimate of State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State —▽—

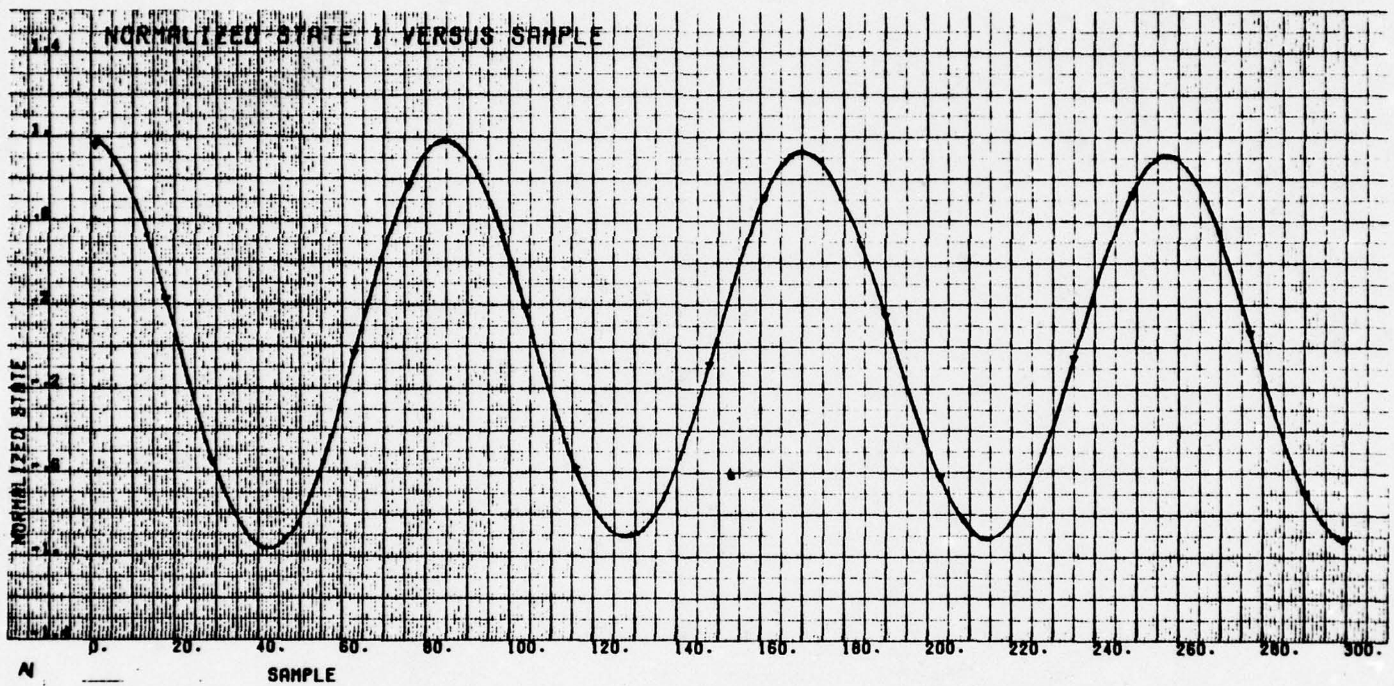

Figure 4. Adaptive Filter Estimate of State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Estimate Error ——

Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

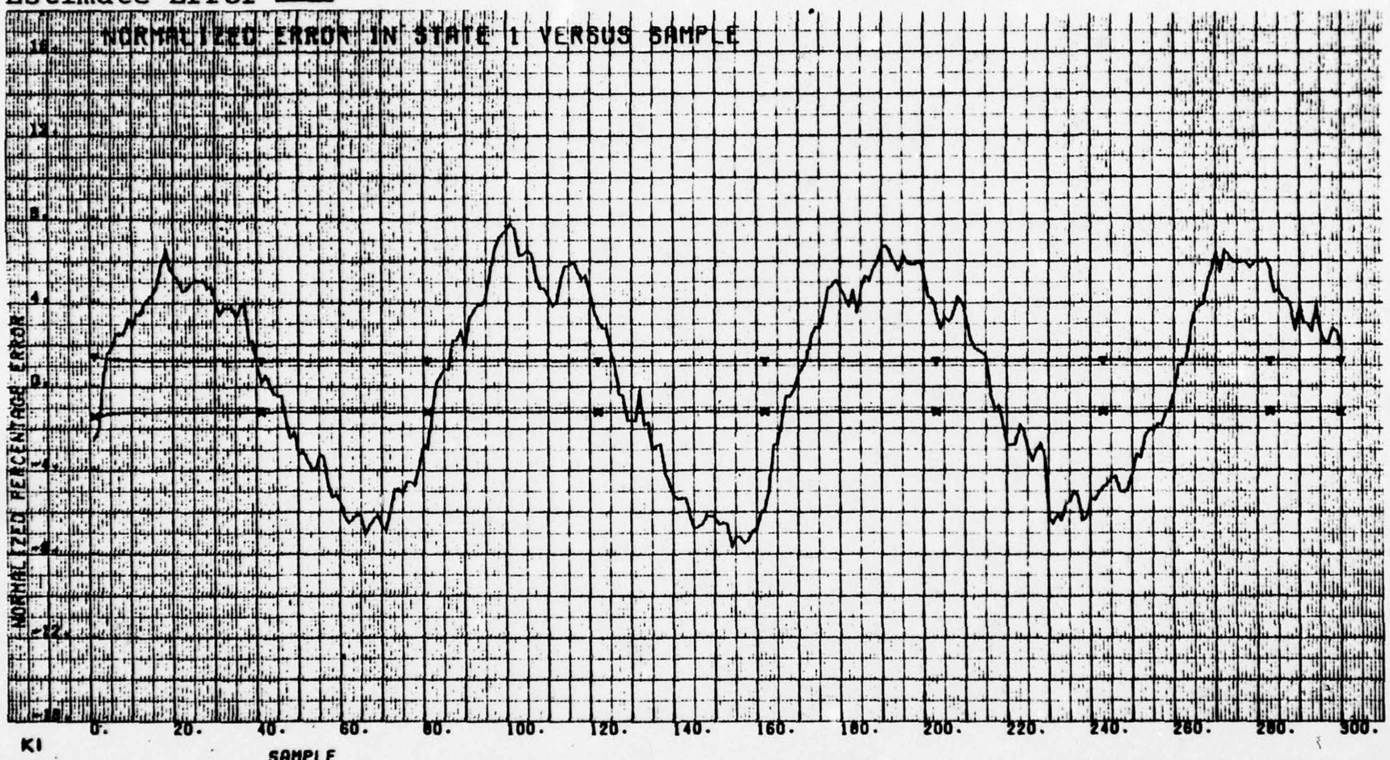

Figure 5. Kalman Filter Estimate Error in State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Estimate Error ——

Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

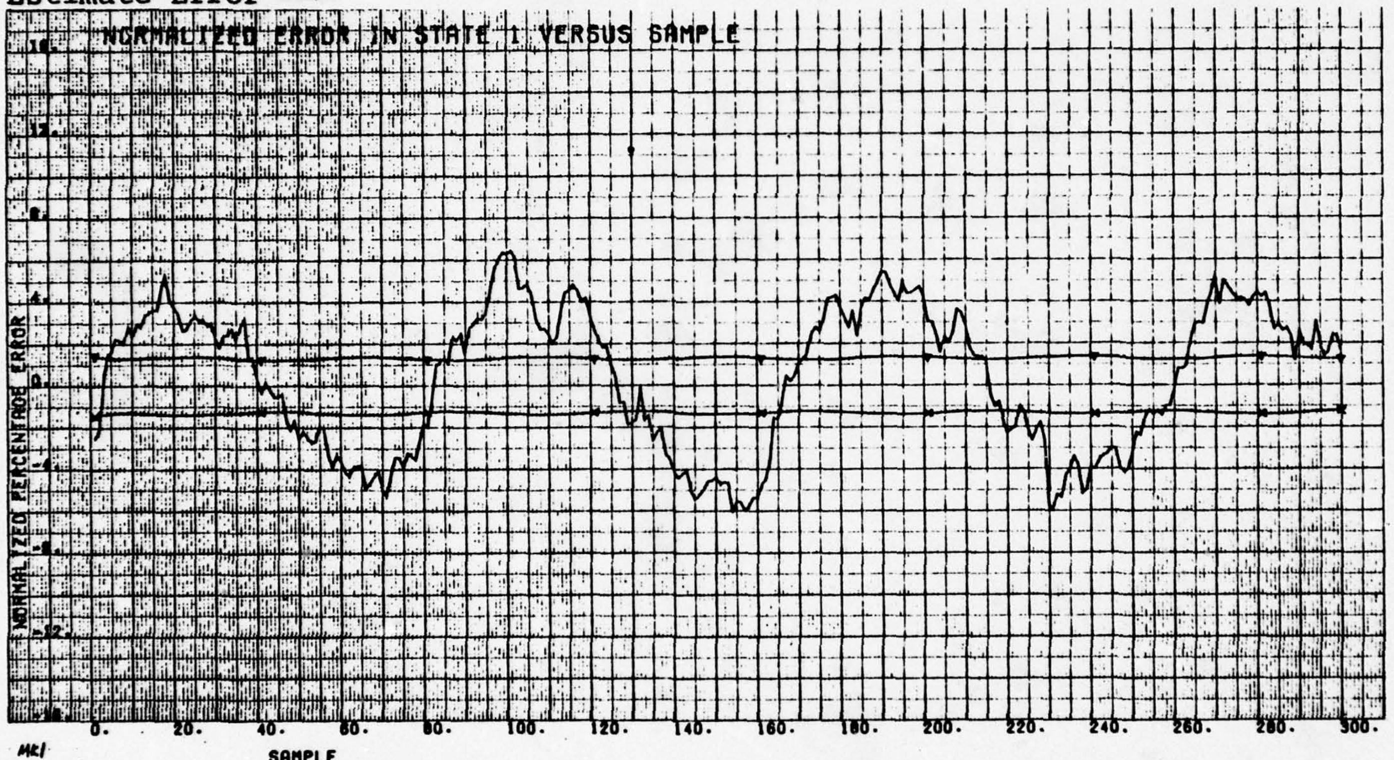

Figure 6. Modified Kalman Filter Estimate Error in State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Estimate Error ——

Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

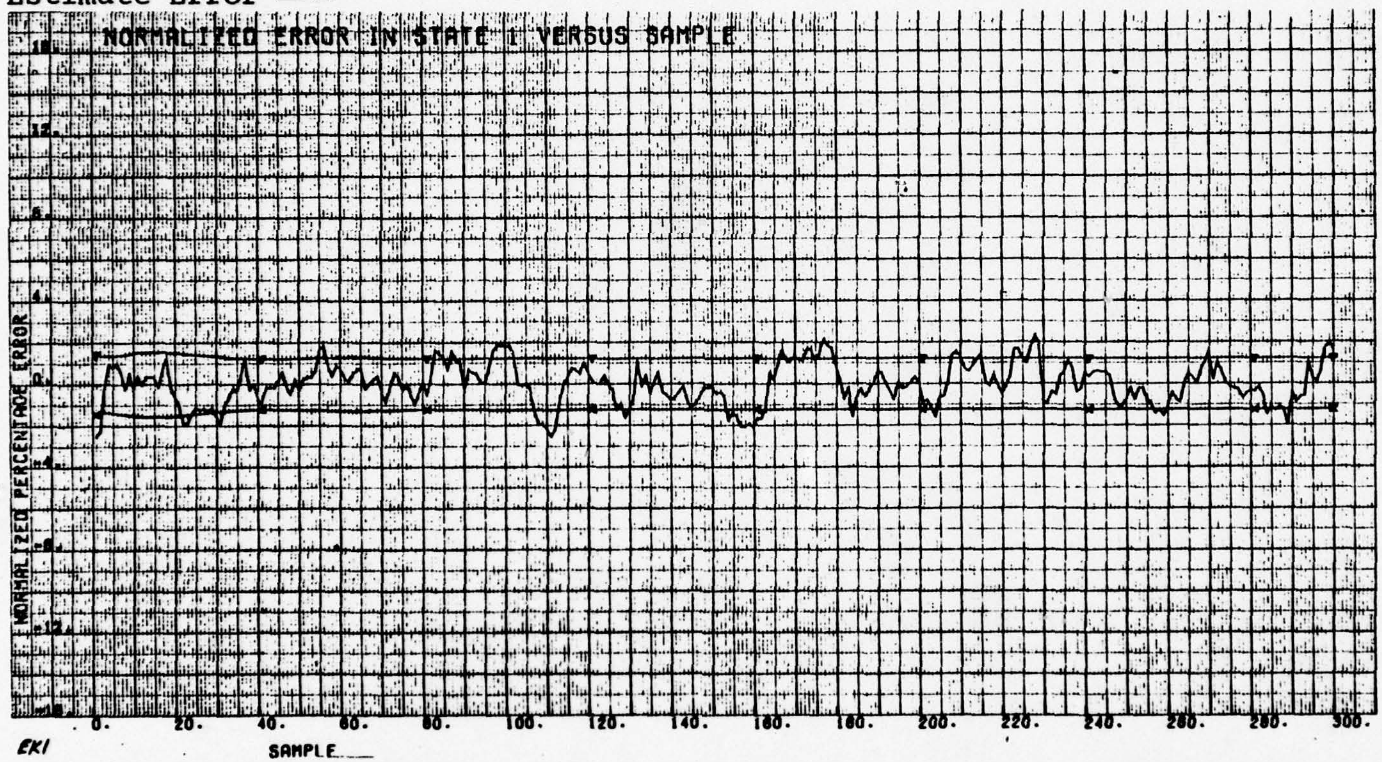

Figure 7. Extended Kalman Filter Estimate Error in State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.66 rad/sec Estimate Error ——
Initial Estimate - 0.63 rad/sec Plus Estimate Standard Deviation
Normalizing Constant - 0.66 rad/sec Minus Estimate Standard Deviation ——
Initial State Covariance - 0.0001 $(\mathrm{rad/sec})^2$
System Noise Covariance - 0.00001 $(\mathrm{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\mathrm{rad/sec})^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

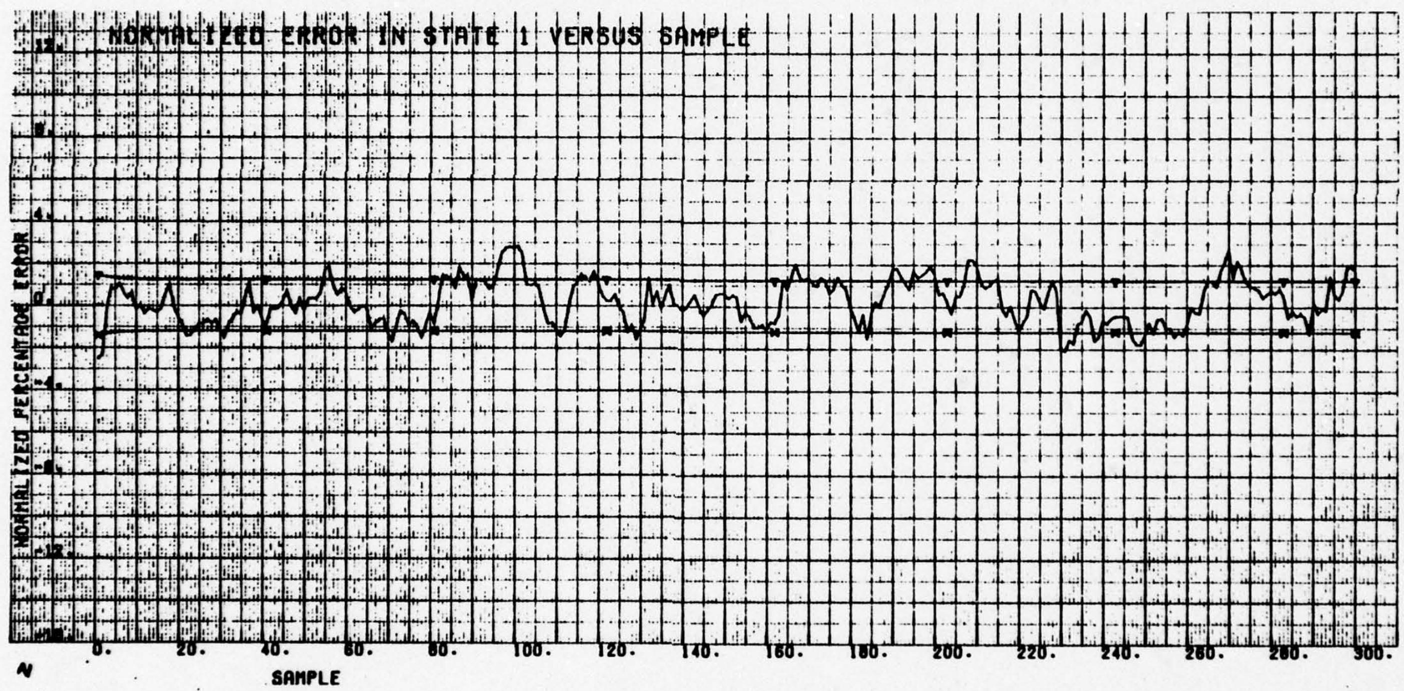

Figure 8. Adaptive Filter Estimate Error in State 1 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State —∇—

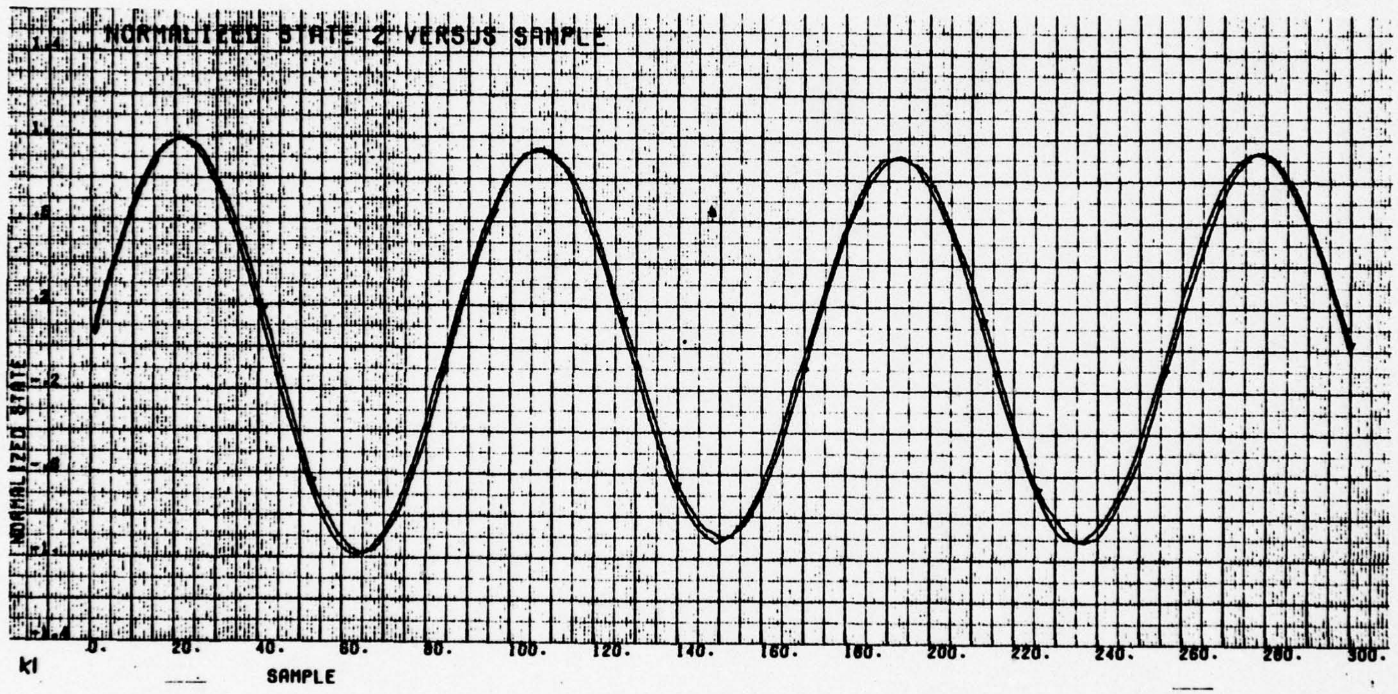

Figure 9. Kalman Filter Estimate of State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

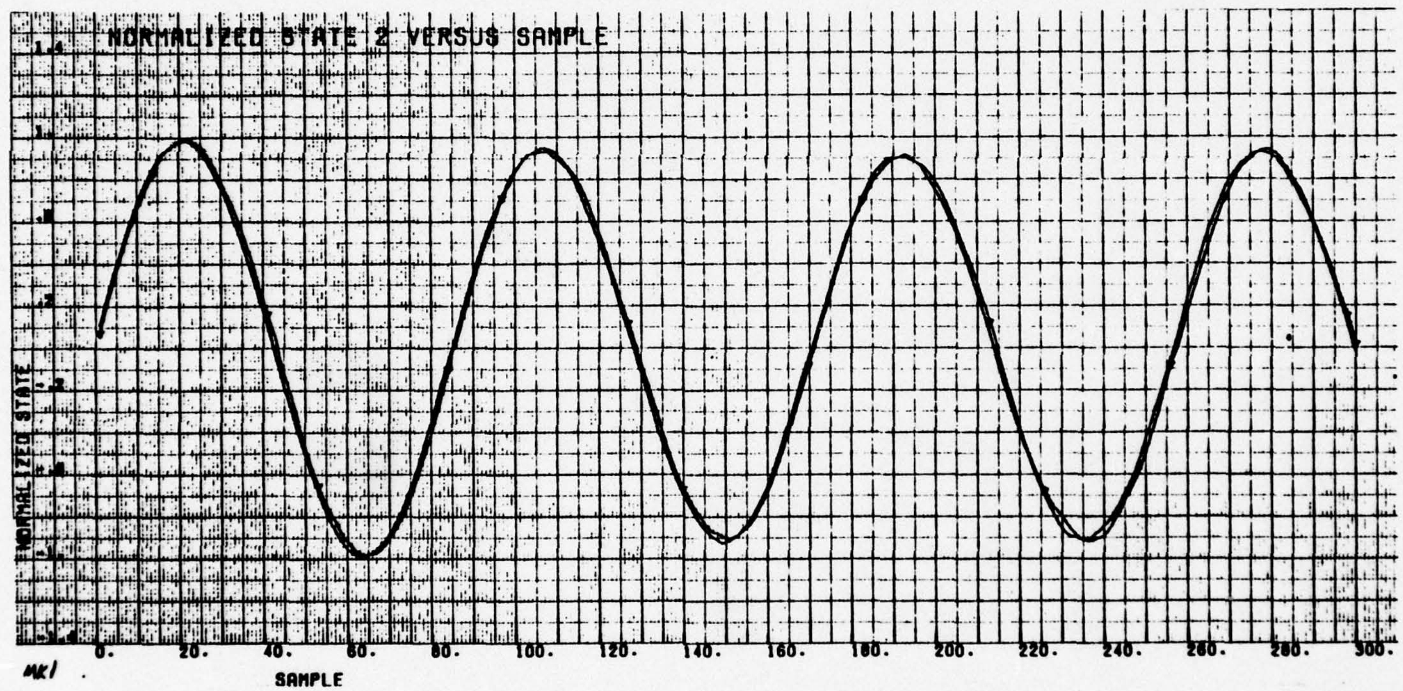

Figure 10. Modified Kalman Filter Estimate of State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

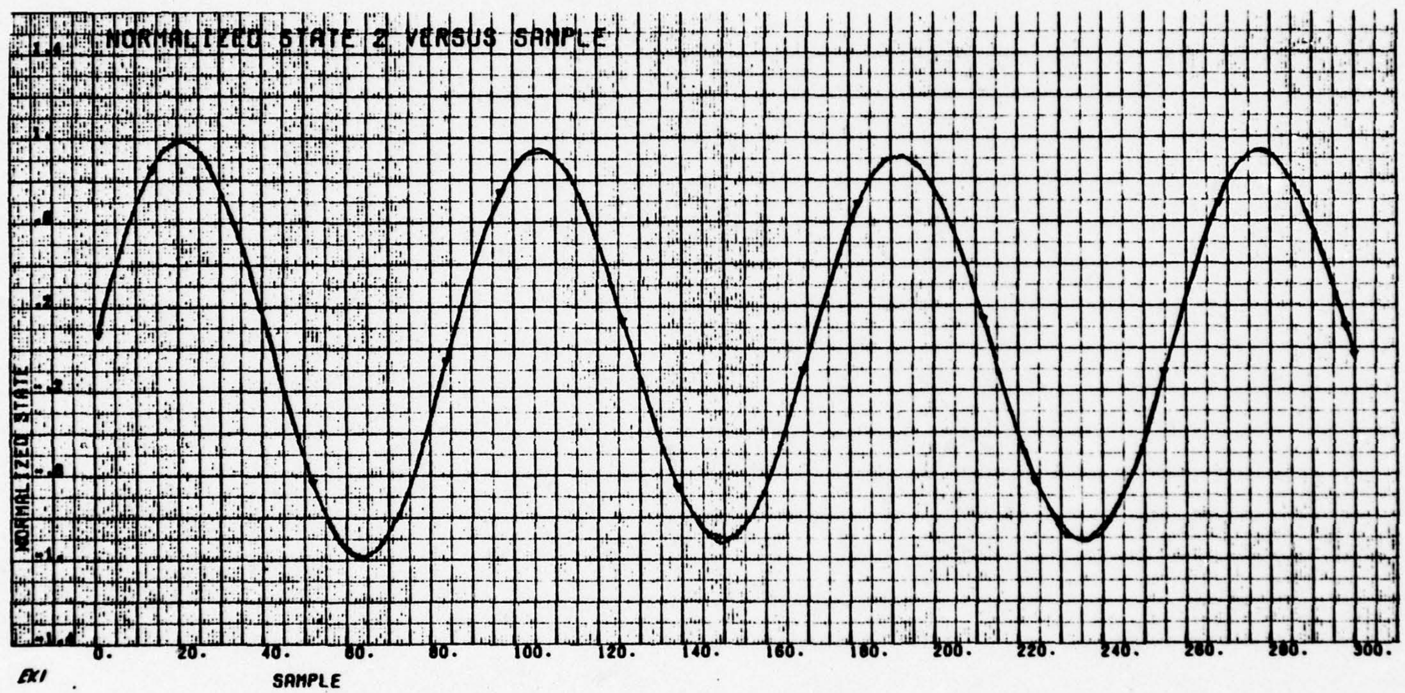

Figure 11. Extended Kalman Filter Estimate of State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State —⊽—

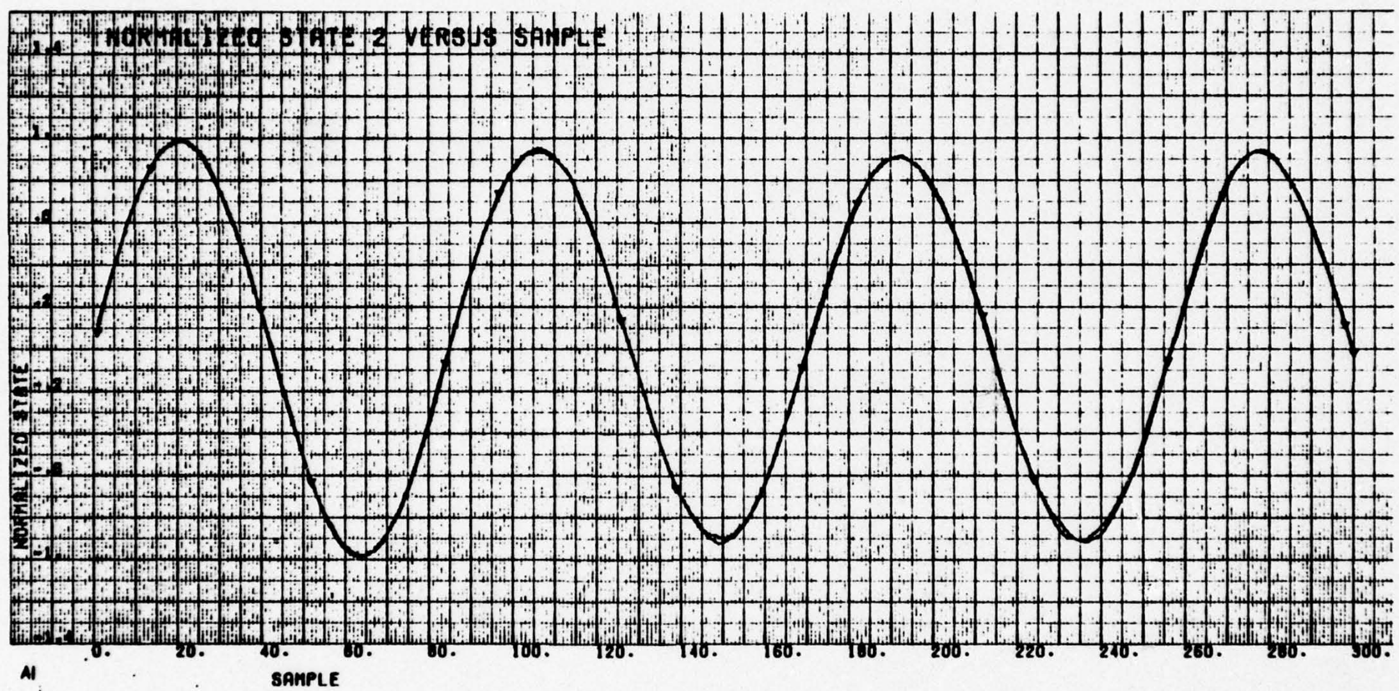

Figure 12. Adaptive Filter Estimate of State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

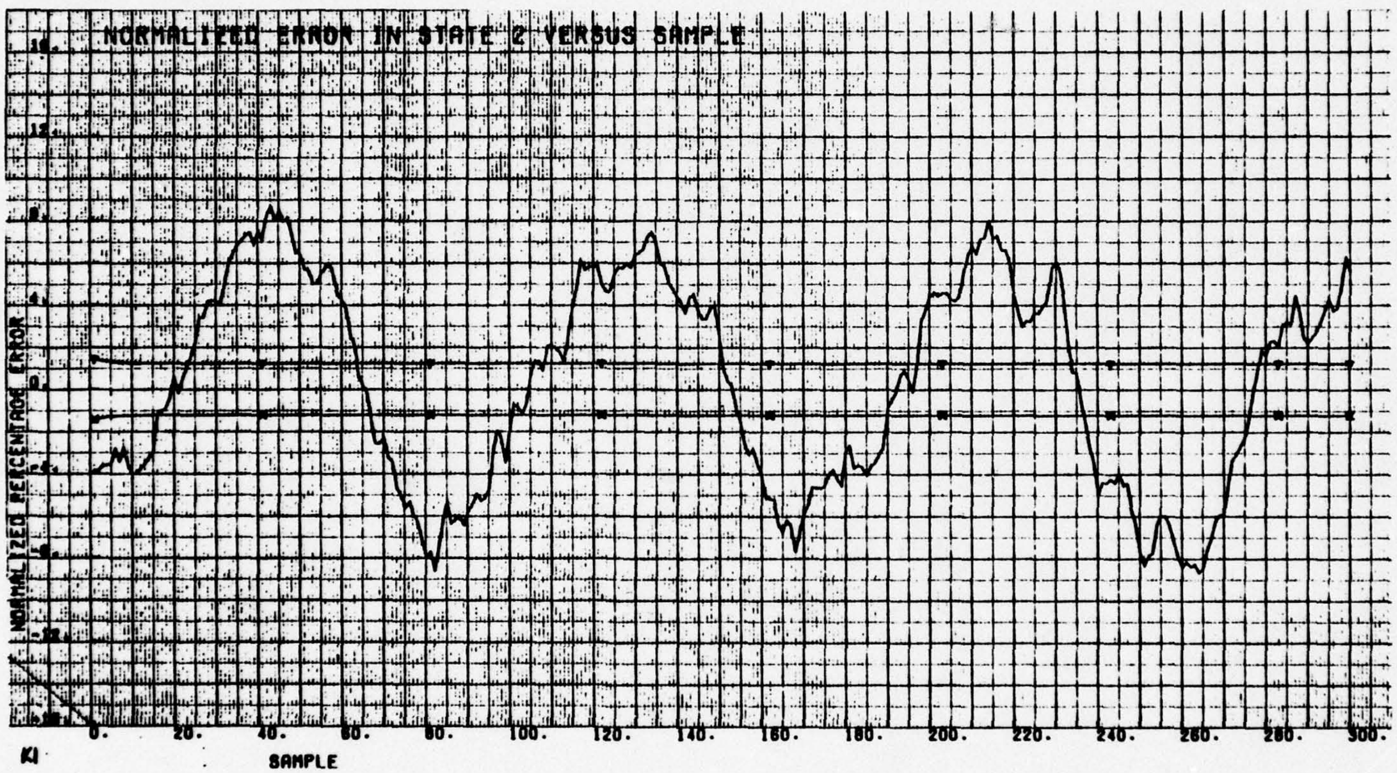

Figure 13. Kalman Filter Estimate Error in State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

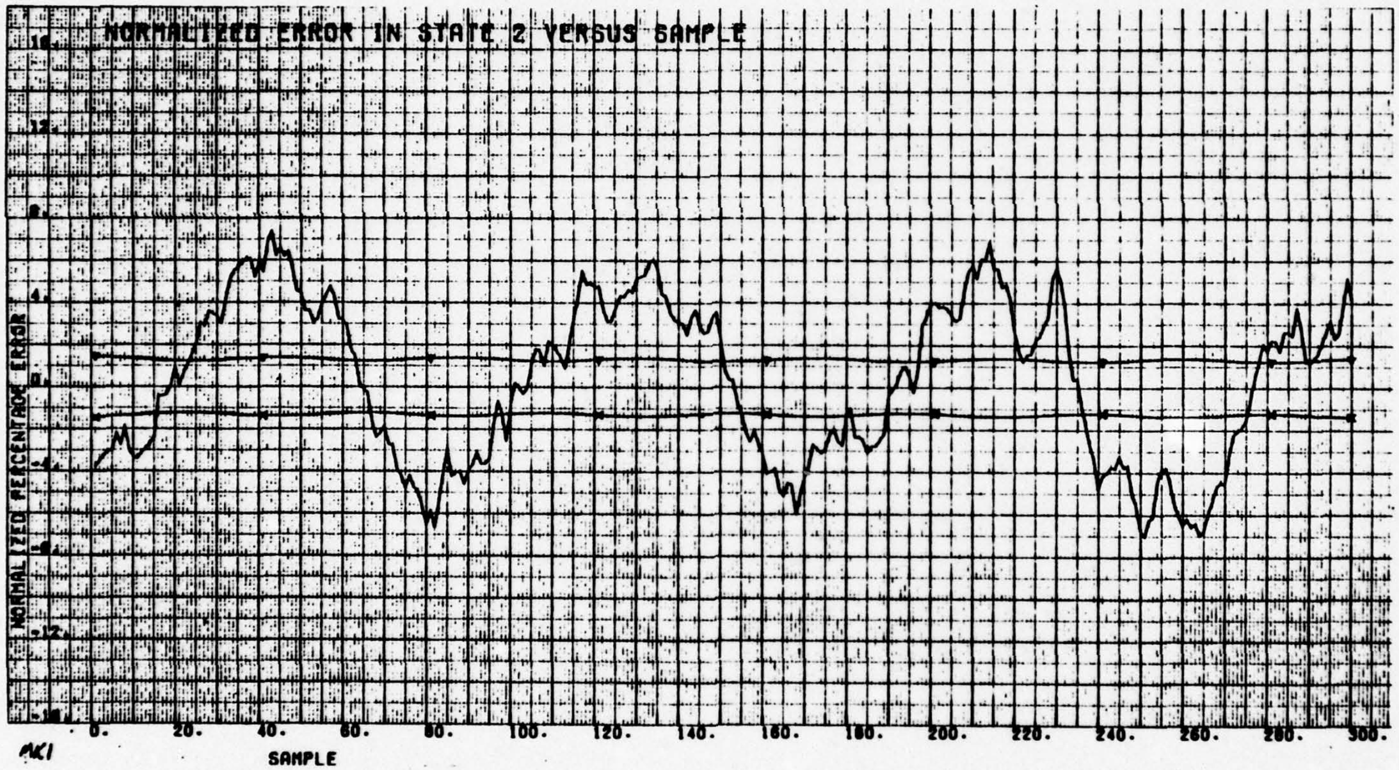

Figure 14. Modified Kalman Filter Estimate Error in State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.3 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

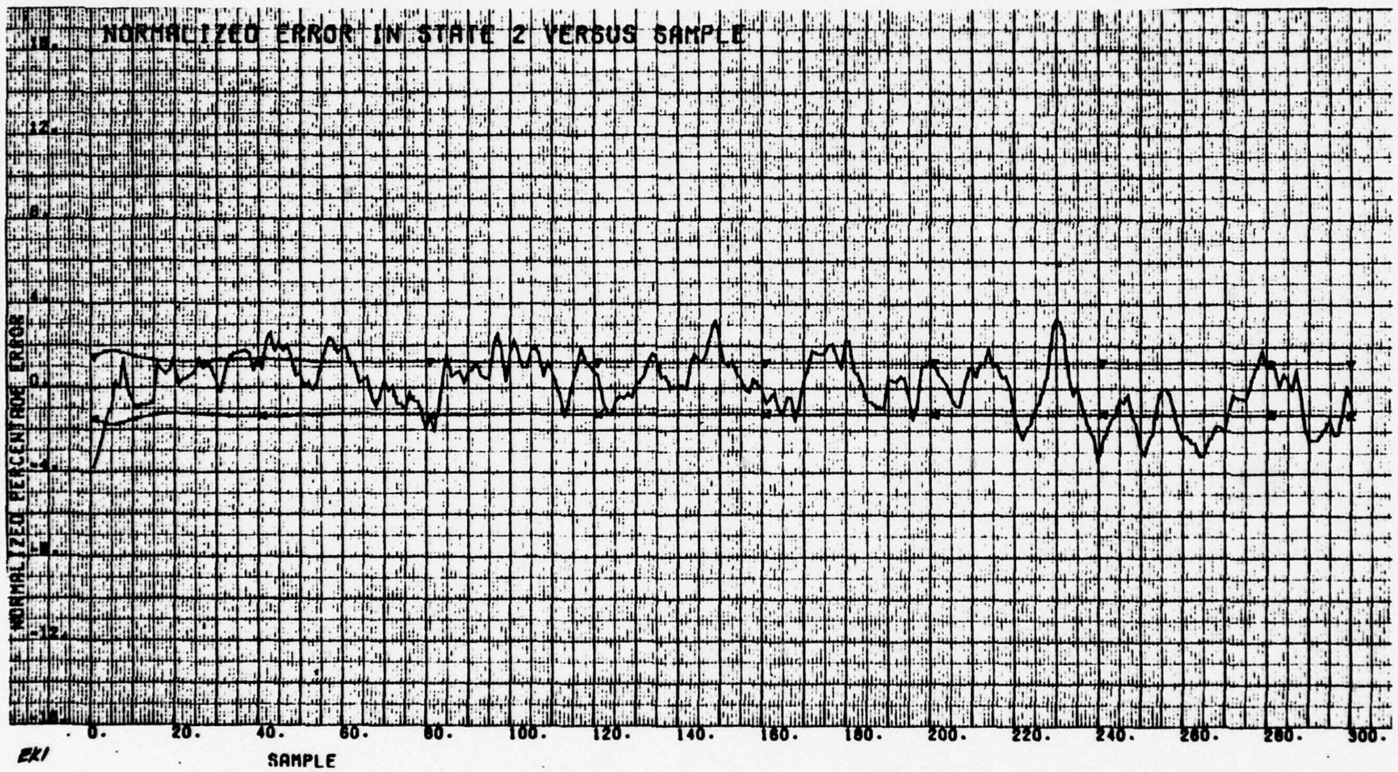

Figure 15. Extended Kalman Filter Estimate Error in State 2 Determined with a Parameter Error in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

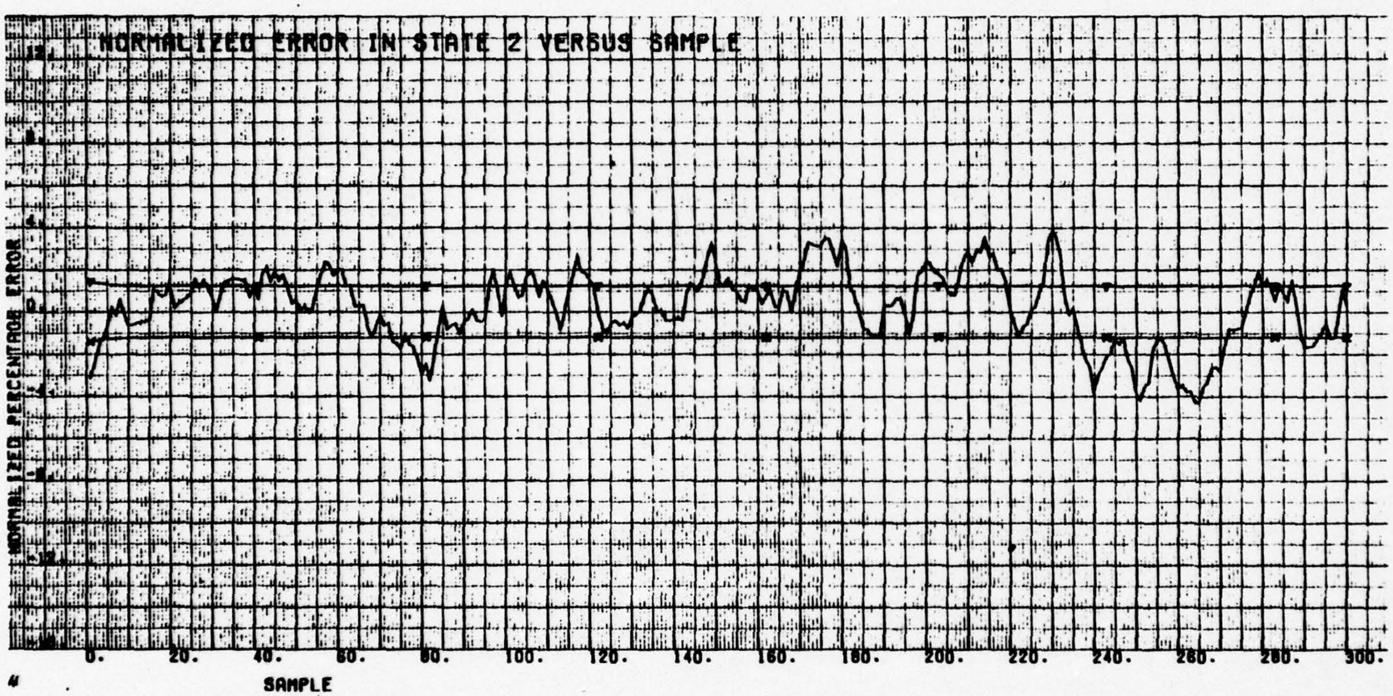

Figure 16. Adaptive Filter Estimate Error in State 2 Determined with a Parameter Error in the System Dynamics.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
True Value ——
Estimated Value

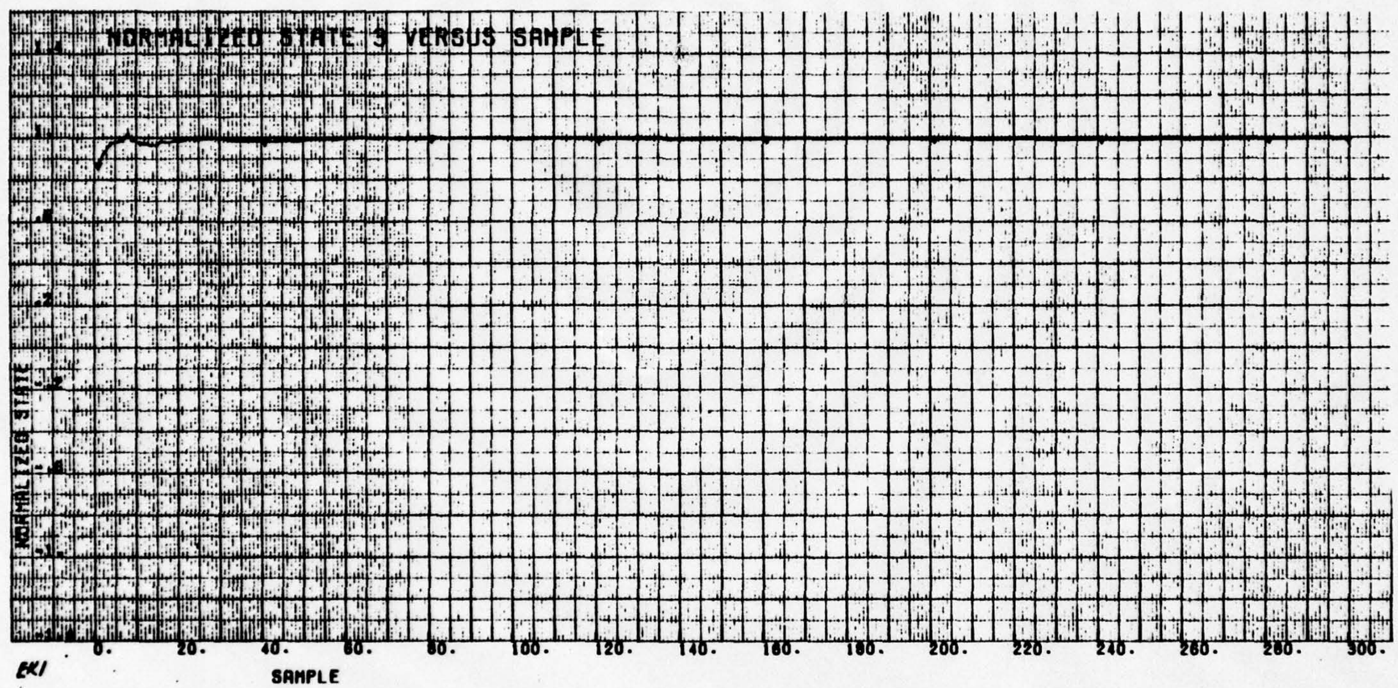

Figure 17. Extended Kalman Filter Estimate of the Uncertain Parameter in the System Dynamics.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True Value ——
Estimated Value

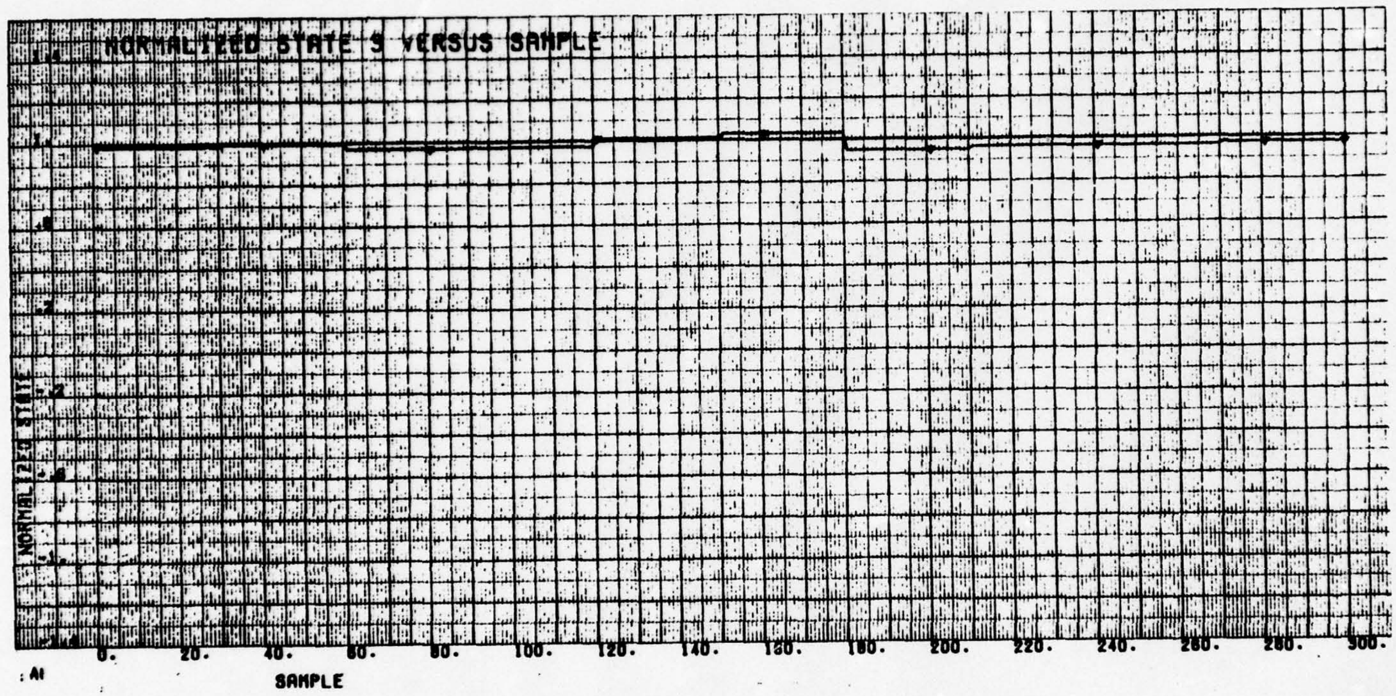

Figure 18. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation —▽—
Minus Estimate Standard Deviation —X—

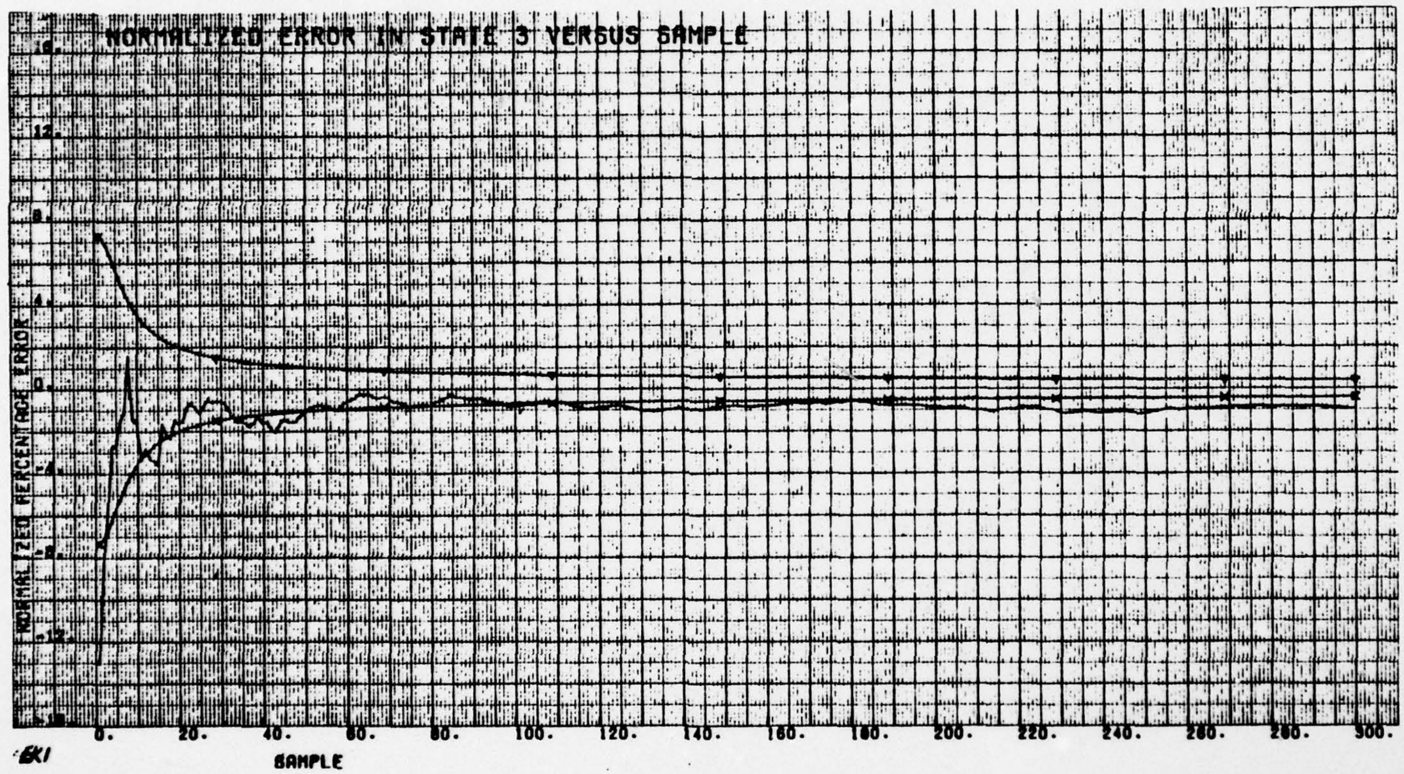

Figure 19. Extended Kalman Filter Estimate Error in the Uncertain Parameter in the System Dynamics.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

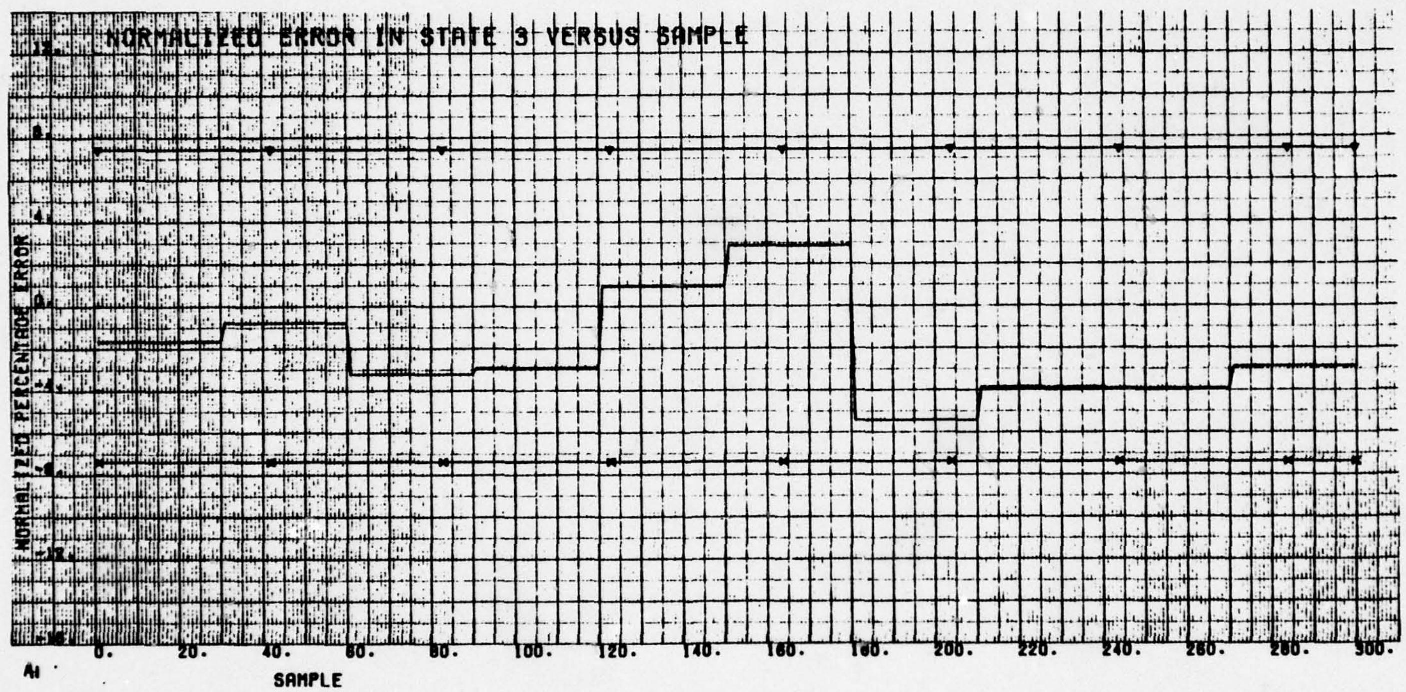

Figure 20. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

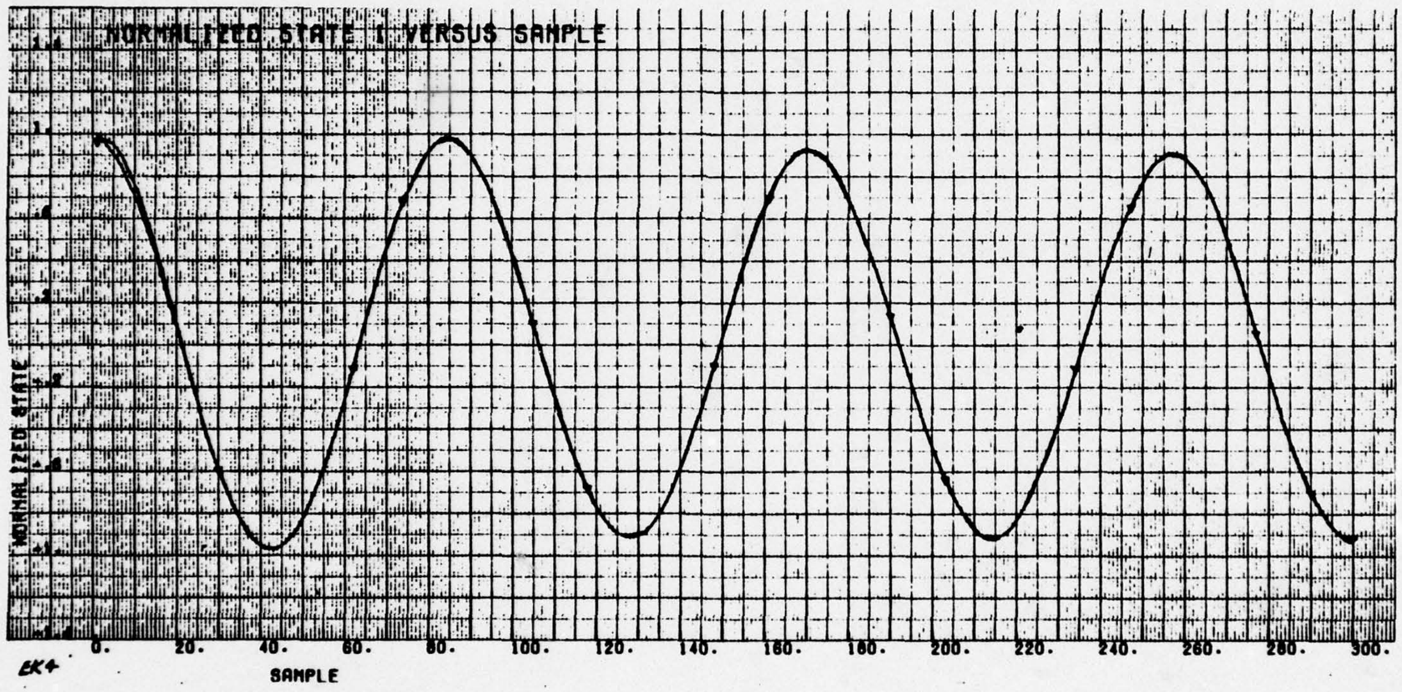

Figure 21. Extended Kalman Filter Estimate of State 1 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State —▽—

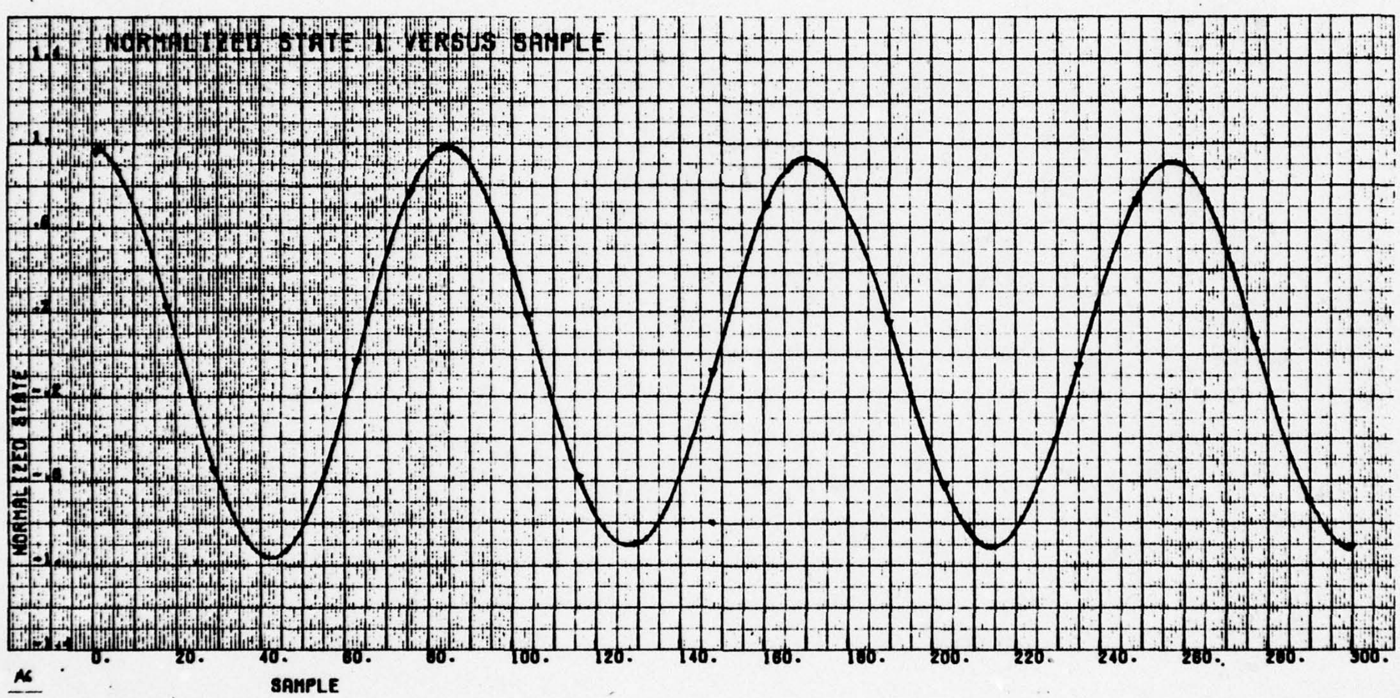

Figure 22. Adaptive Filter Estimate of State 1 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

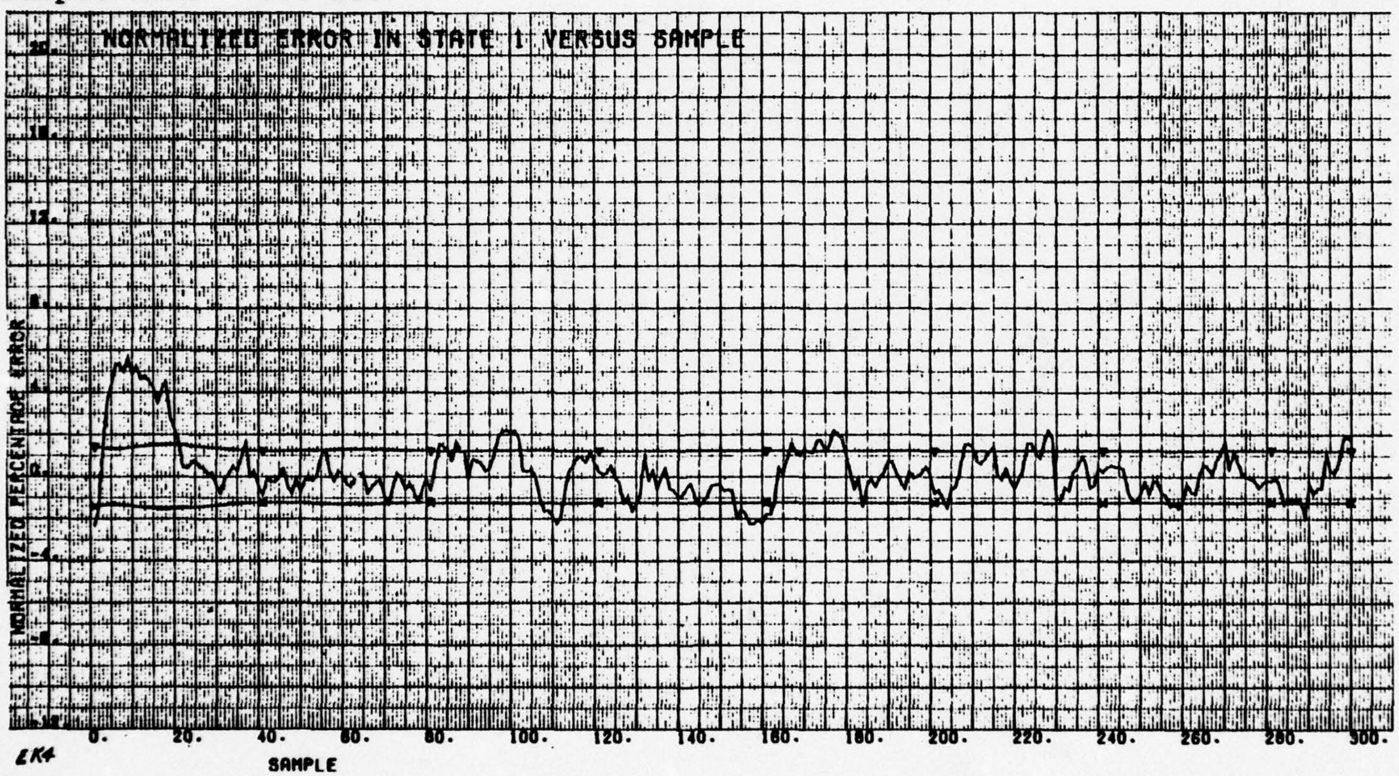

Figure 23. Extended Kalman Filter Estimate Error in State 1 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Estimate Error ——
Plus Estimate Standard Deviation —▽—
Minus Estimate Standard Deviation —X—

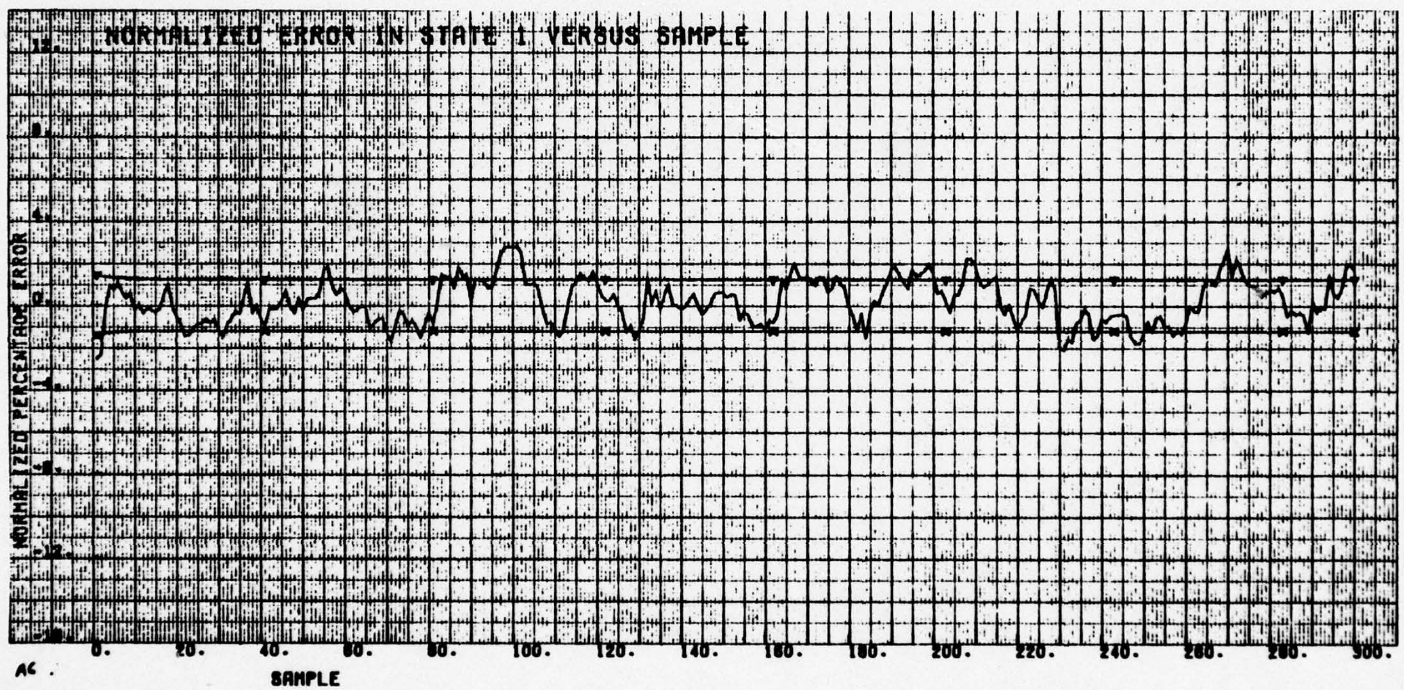

Figure 24. Adaptive Kalman Filter Estimate Error in State 1 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State —▽—

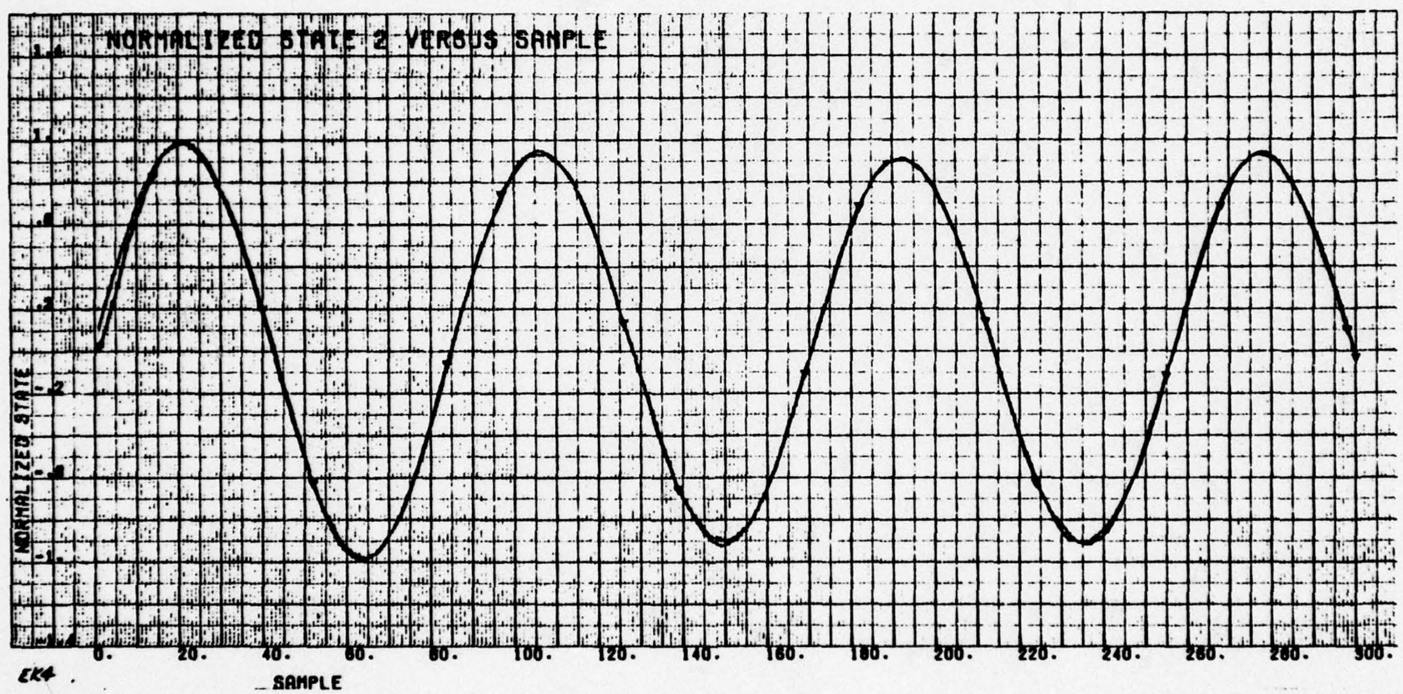

Figure 25. Extended Kalman Filter Estimate of State 2 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State —▽—

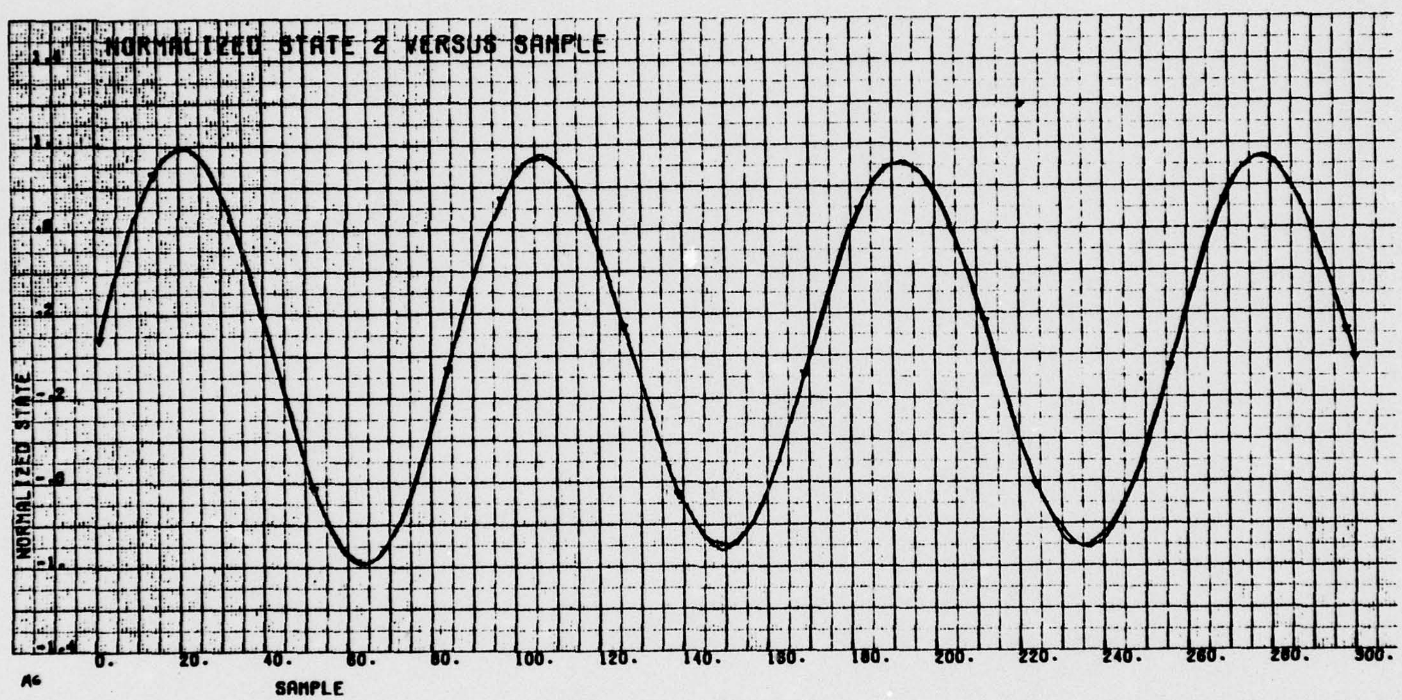

Figure 26. Adaptive Filter Estimate of State 2 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation —▽—
Minus Estimate Standard Deviation ——

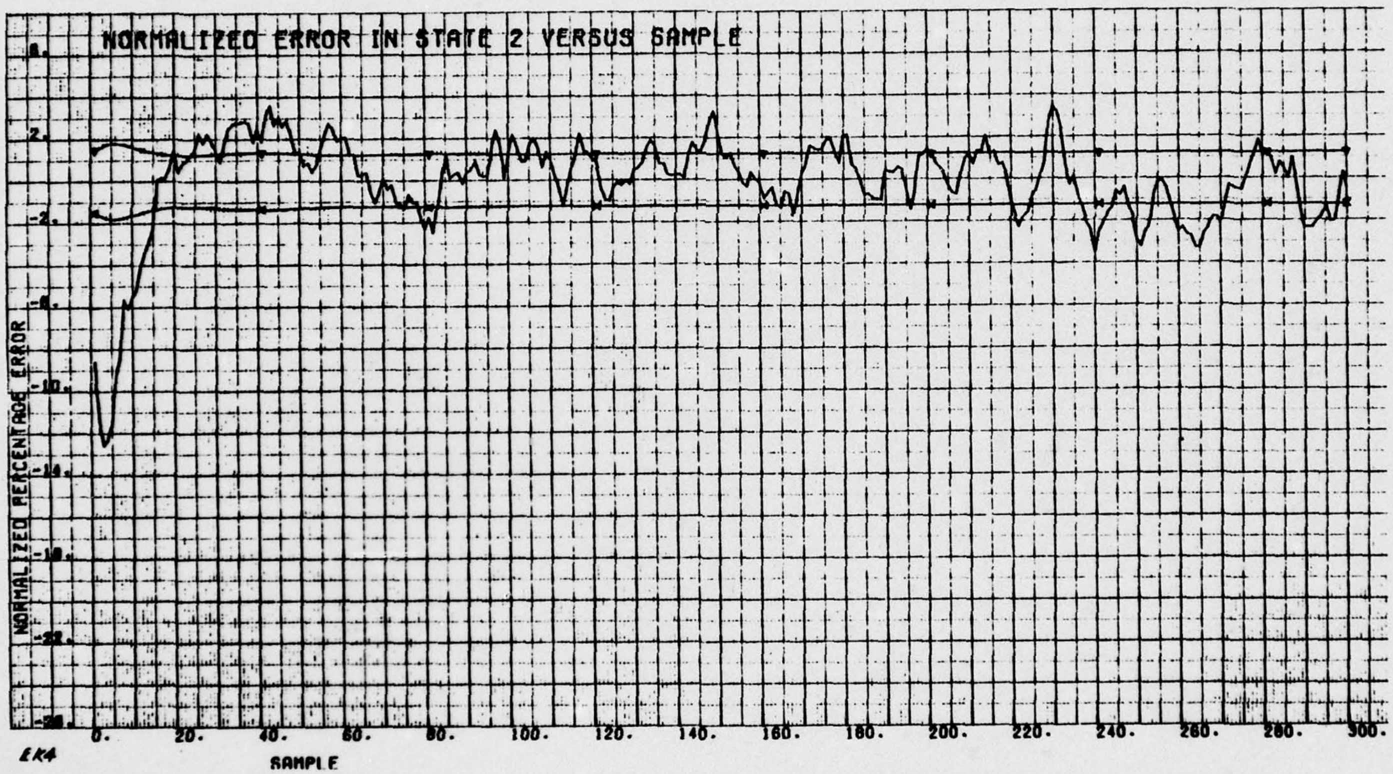

Figure 27. Extended Kalman Filter Estimate Error in State 2 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Initial State - 0.03 rad/sec	Estimate Error ——
Initial Estimate - 0.0 rad/sec	Plus Estimate Standard Deviation
Normalizing Constant - 0.66 rad/sec	Minus Estimate Standard Deviation

Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

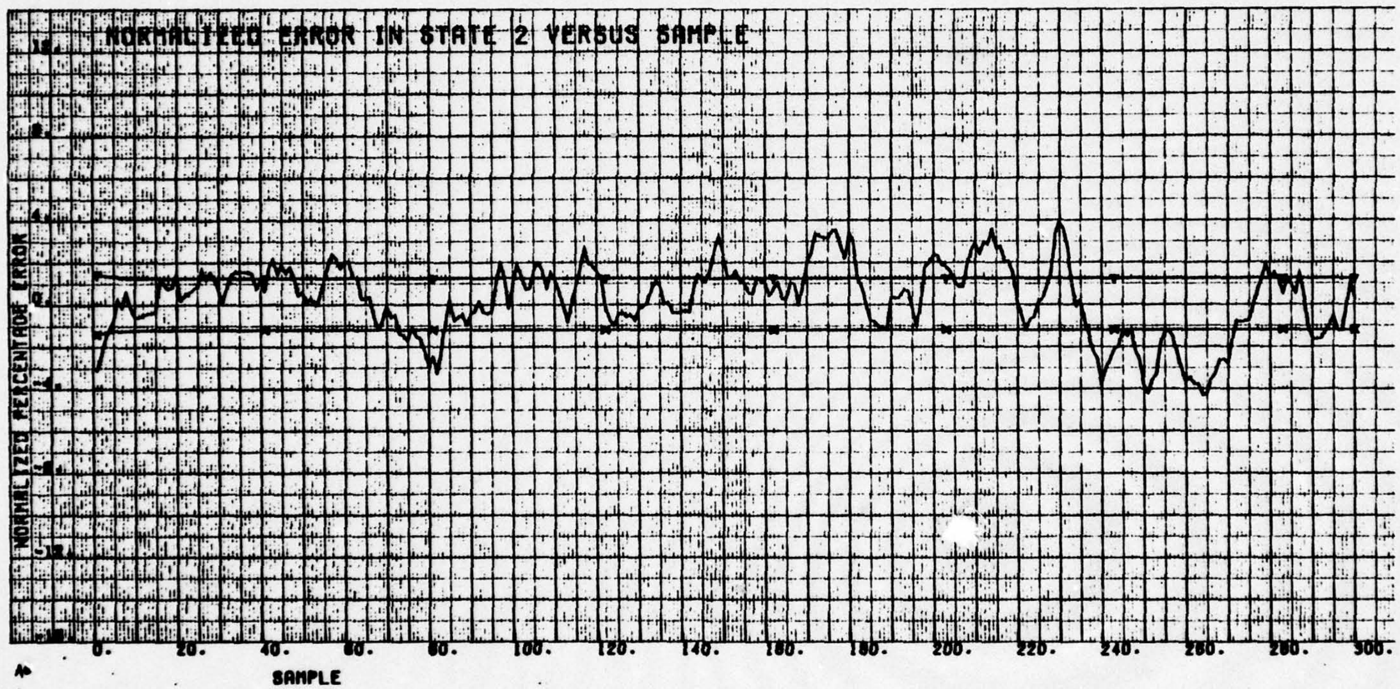

Figure 28. Adaptive Filter Estimate Error in State 2 Determined from a Zero Initial Estimate for the Uncertain Parameter in the System Dynamics.

Parameter Value - 7.4 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
True Value ——
Estimated Value

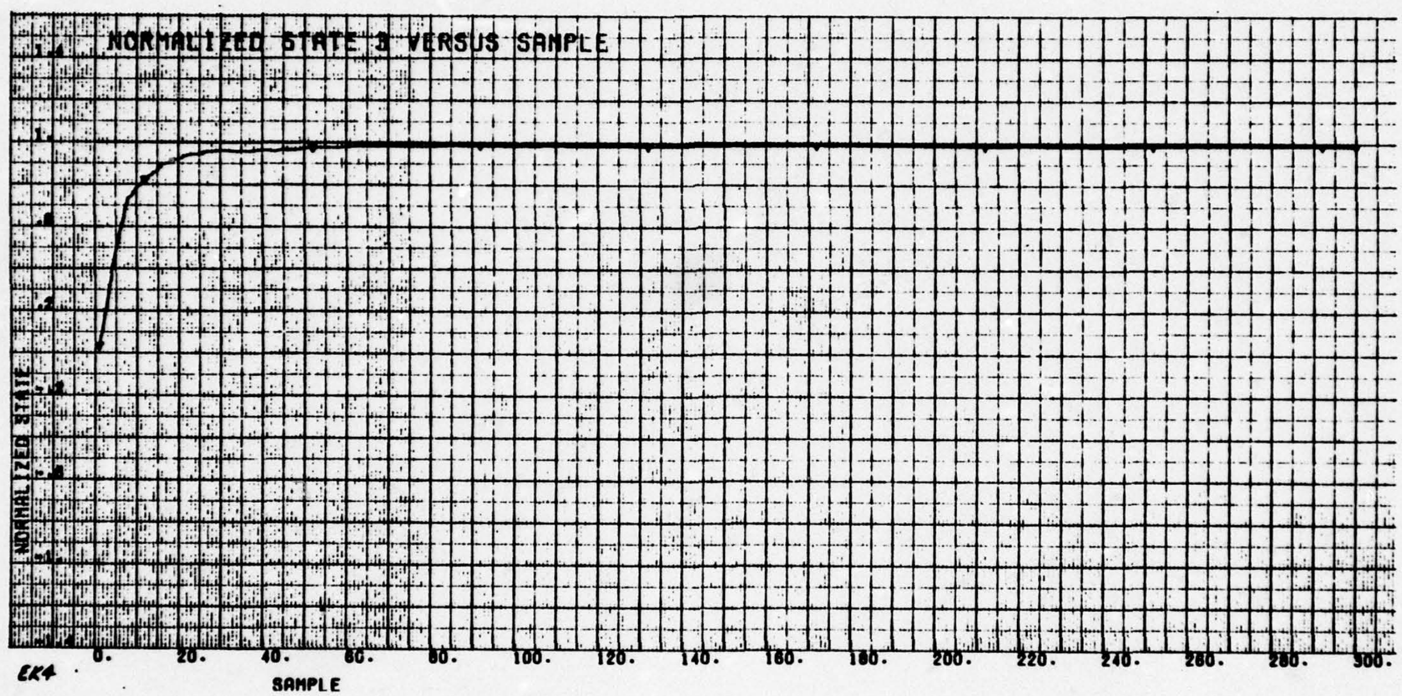

Figure 29. Extended Kalman Filter Estimate of the Uncertain Parameter in the System Dynamics Determined from a Zero Initial Estimate.

Parameter Value - 7.4 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
True Value ——
Estimated Value —▽—

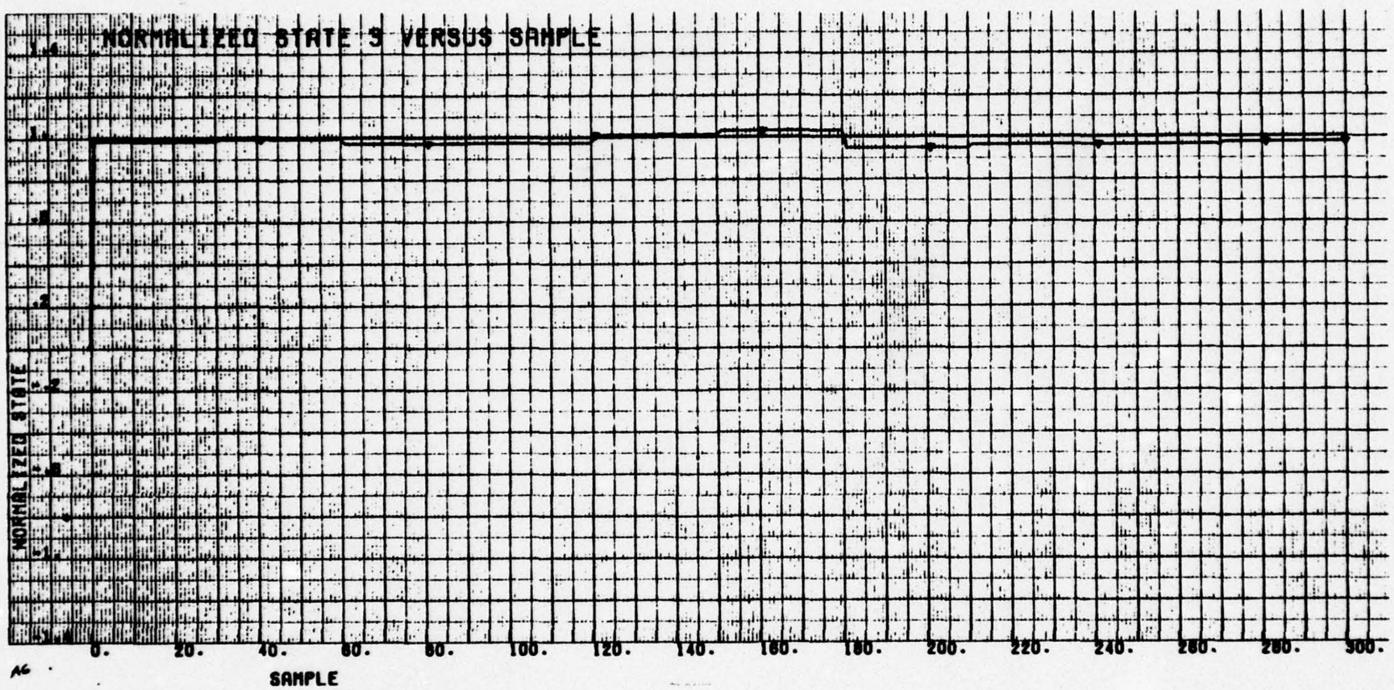

Figure 30. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics Determined from a Zero initial Estimate.

Parameter Value - 7.4 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

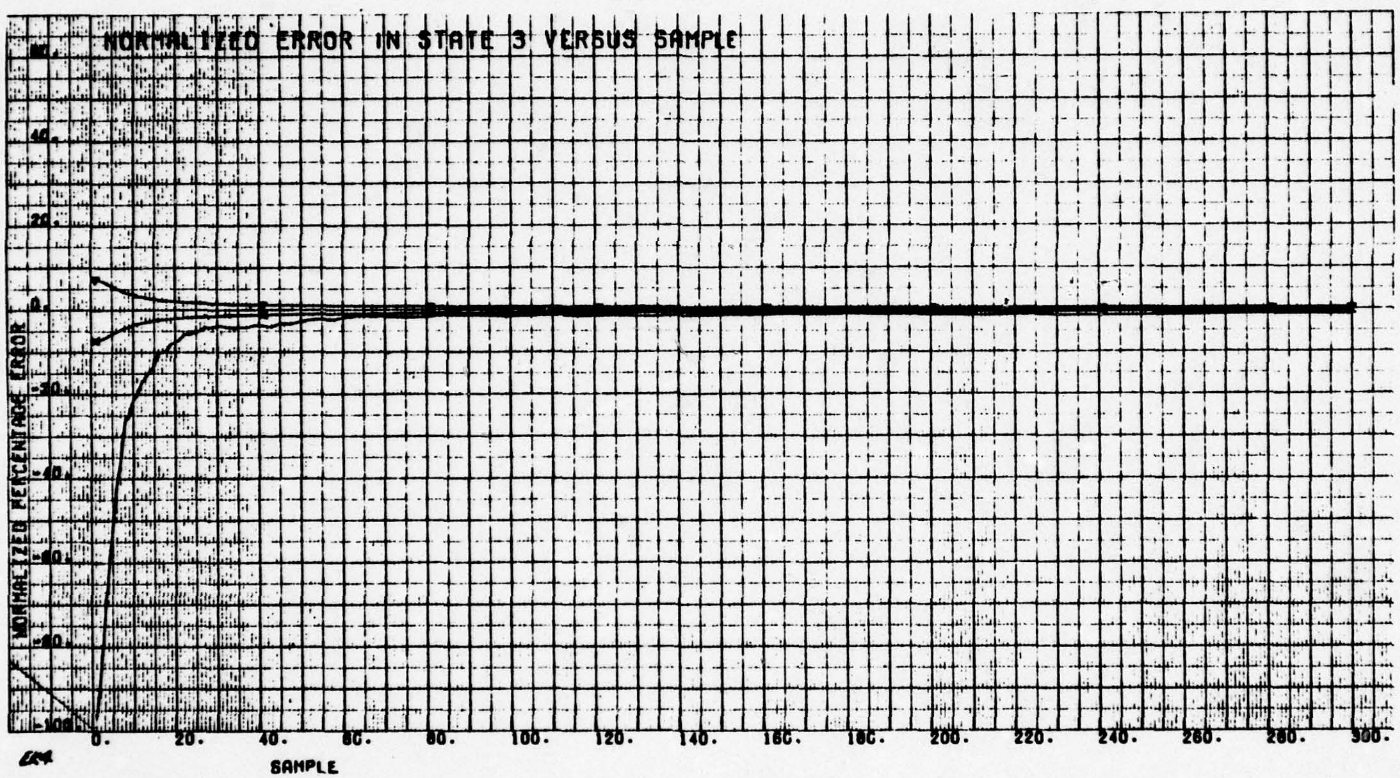

Figure 31. Extended Kalman Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined from a Zero Initial Estimate.

Parameter Value - 7.4 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Estimate Error
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

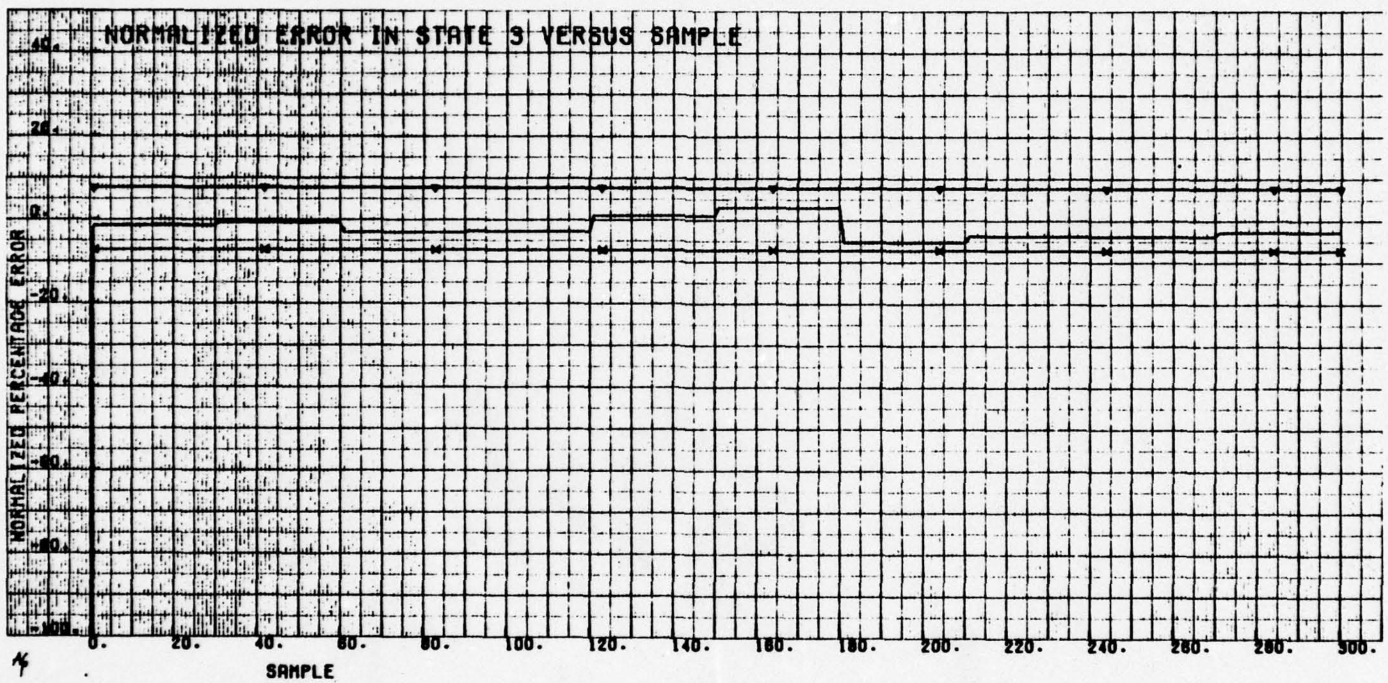

Figure 32. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined from a Zero Initial Estimate.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State -▽-

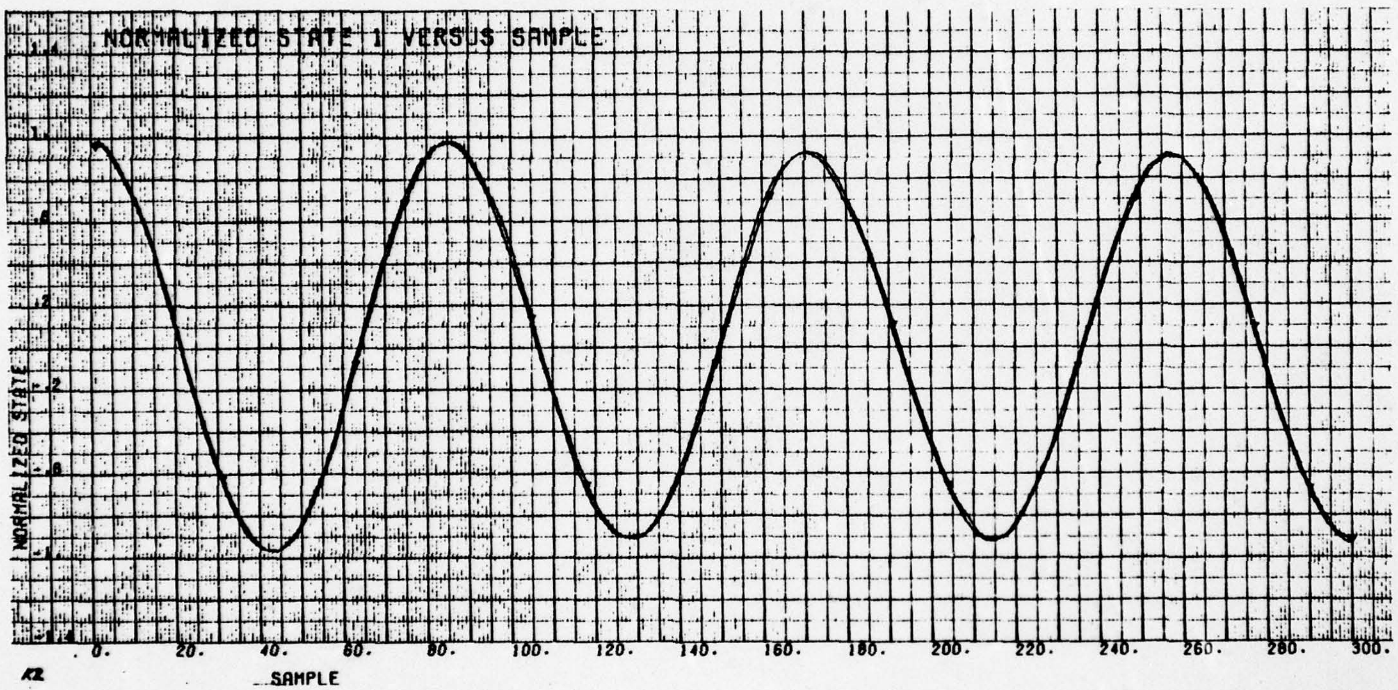

Figure 33. Kalman Filter Estimate of State 1 Determined with Reduced Measurement Noise.

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Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

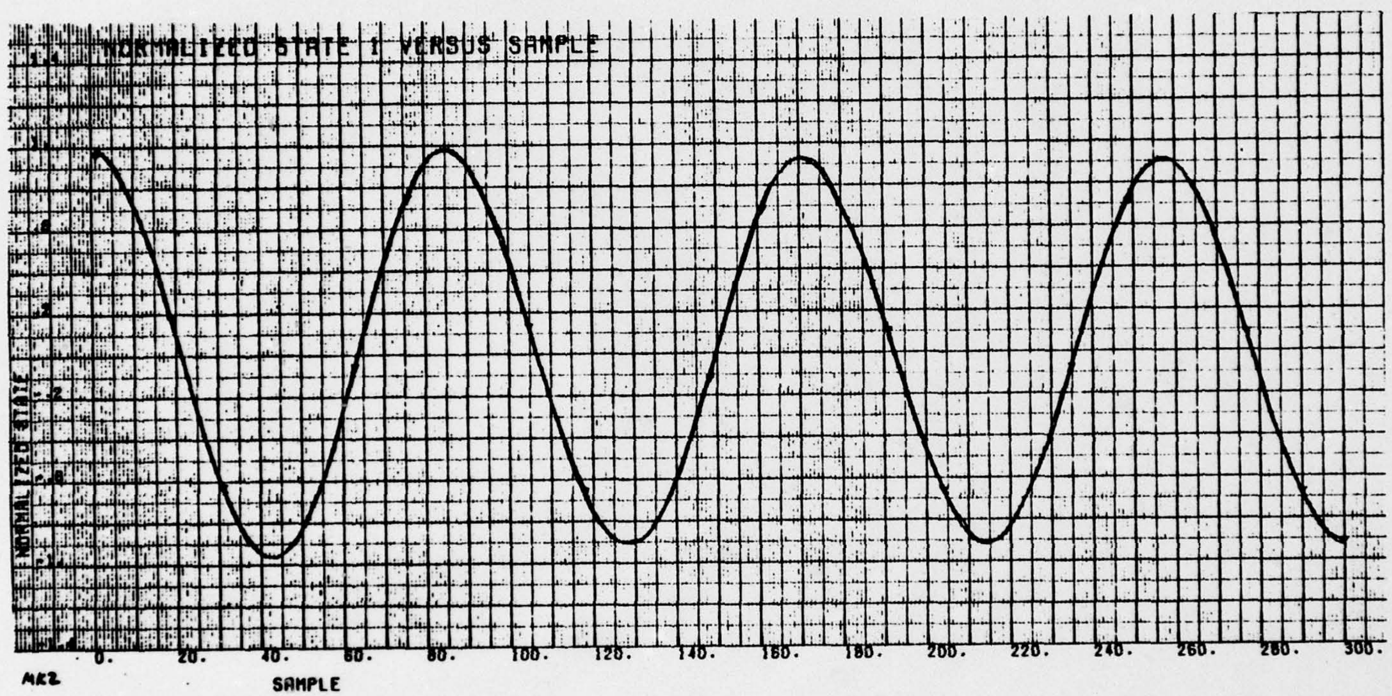

Figure 34. Modified Kalman Filter Estimate of State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

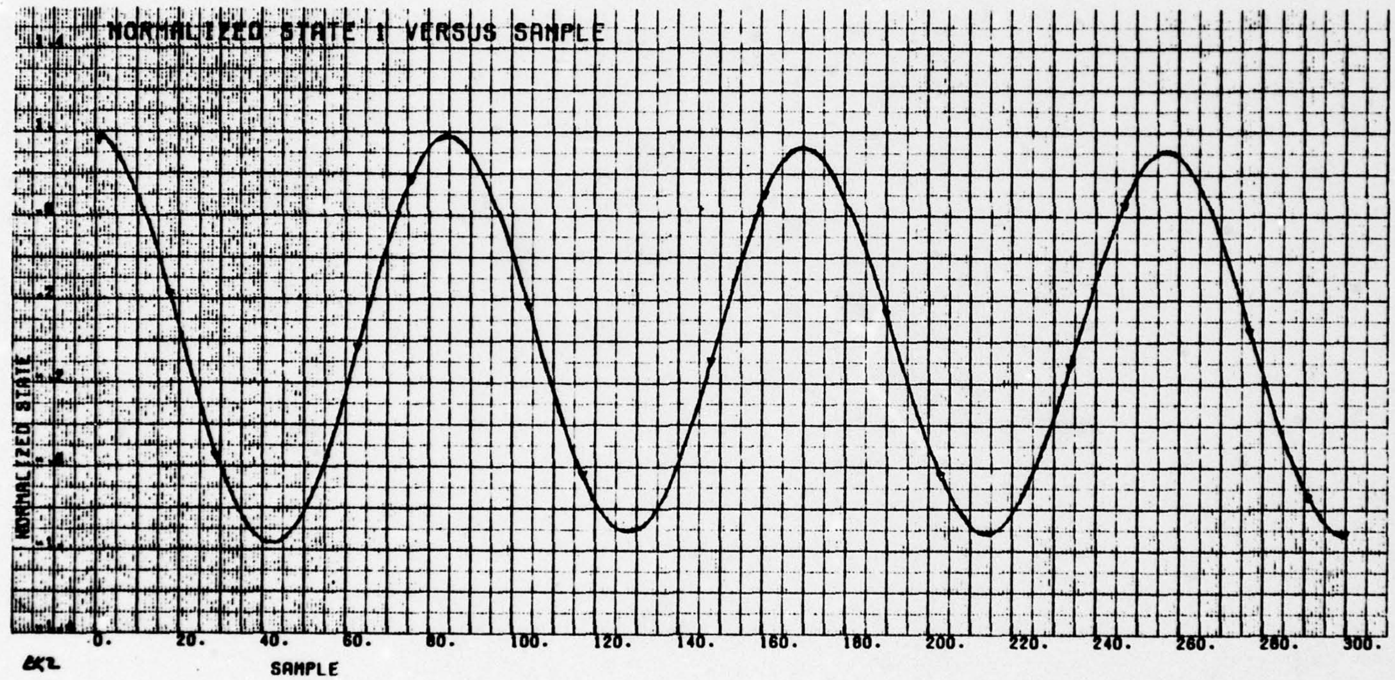

Figure 35. Extended Kalman Filter Estimate of State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
True State ——
Estimated State —▽—

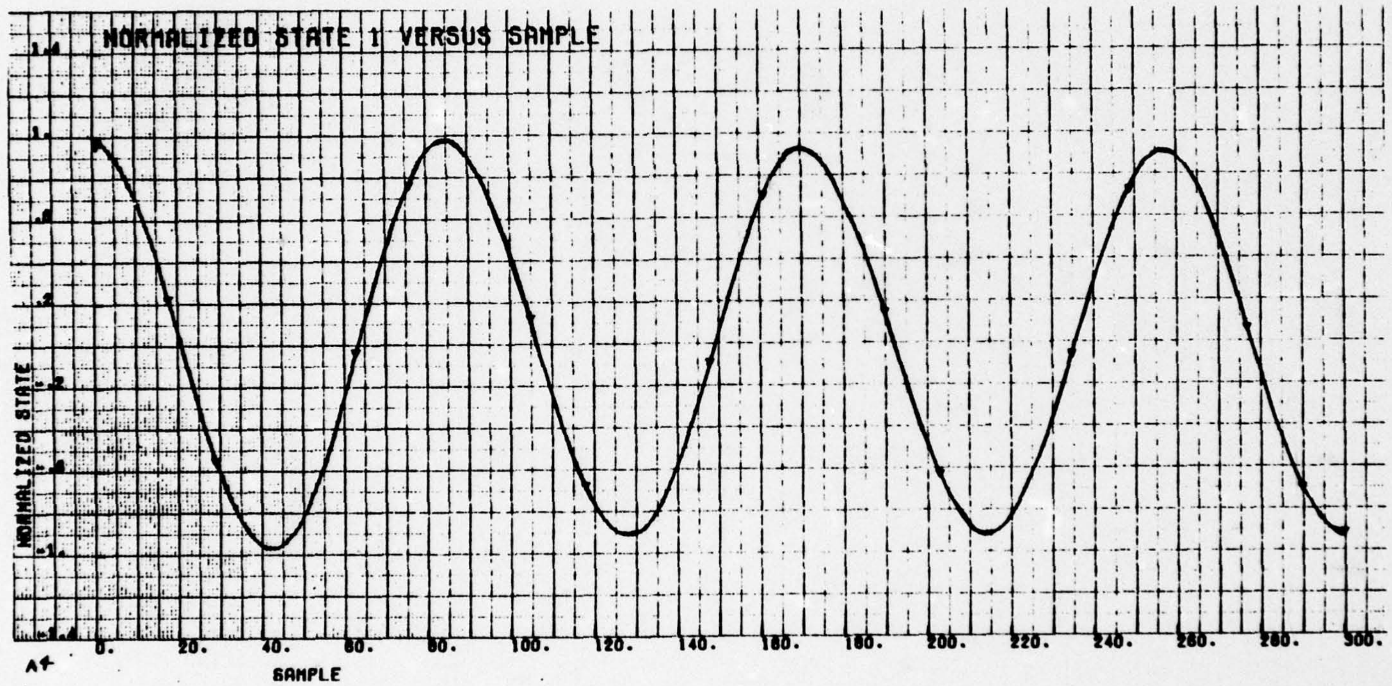

Figure 36. Adaptive Filter Estimate of State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation —◊—
Minus Estimate Standard Deviation —X—

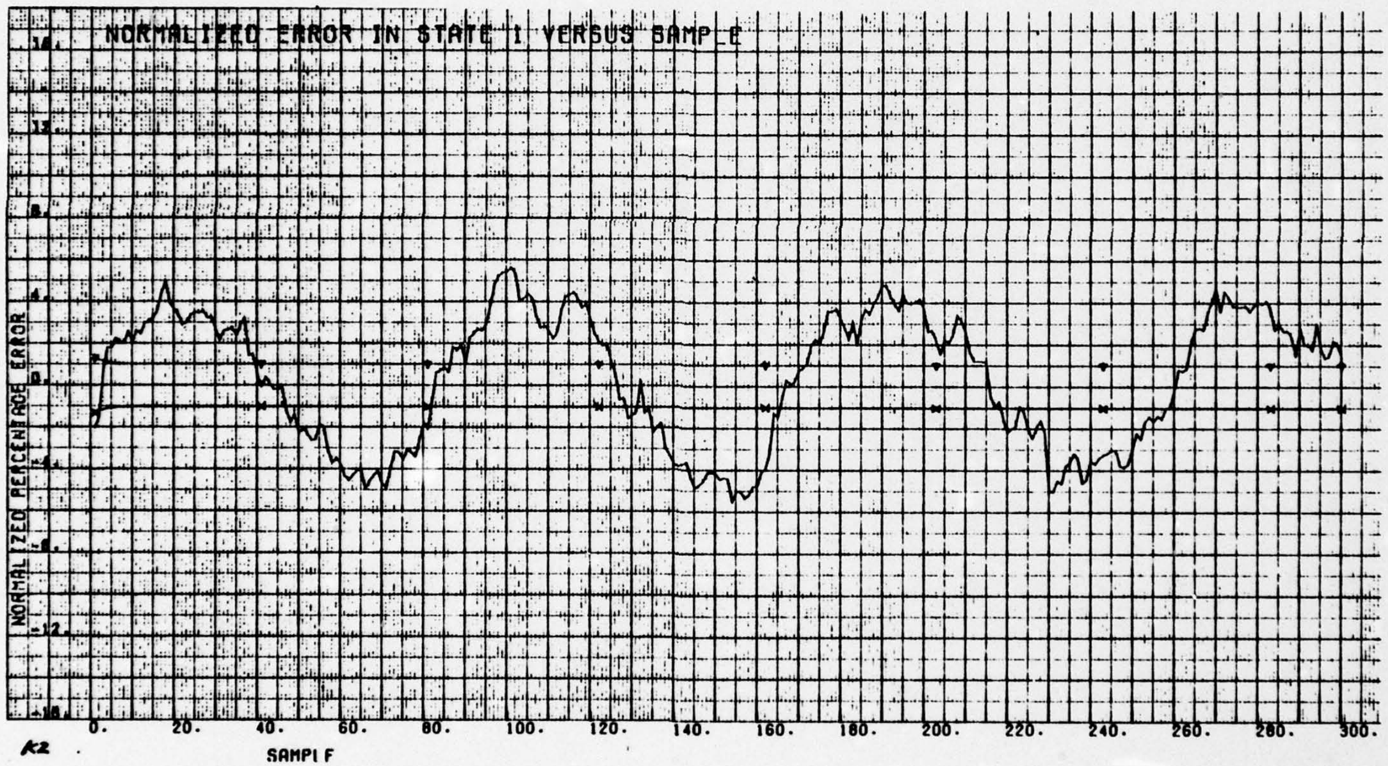

Figure 37. Kalman Filter Estimate Error in State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.00025 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

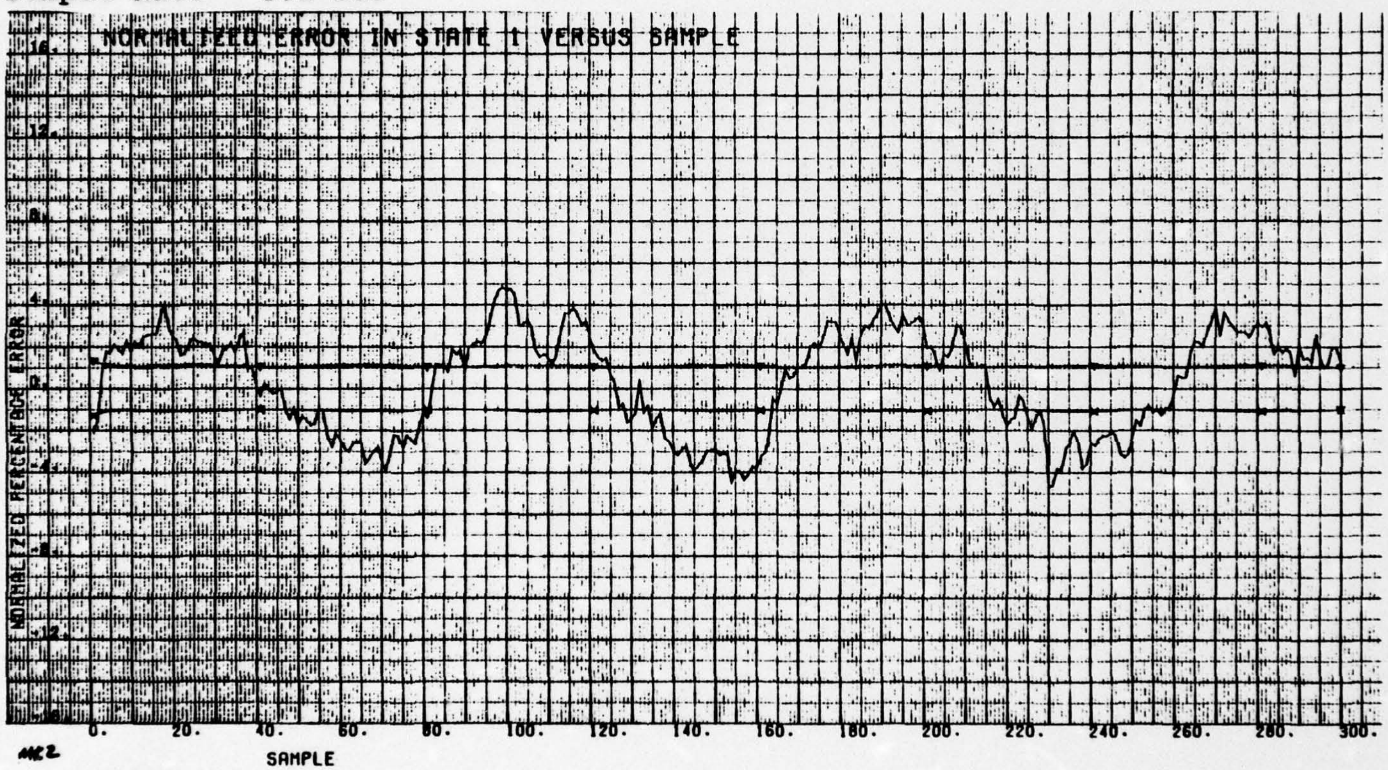

Figure 38. Modified Kalman Filter Estimate Error in State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

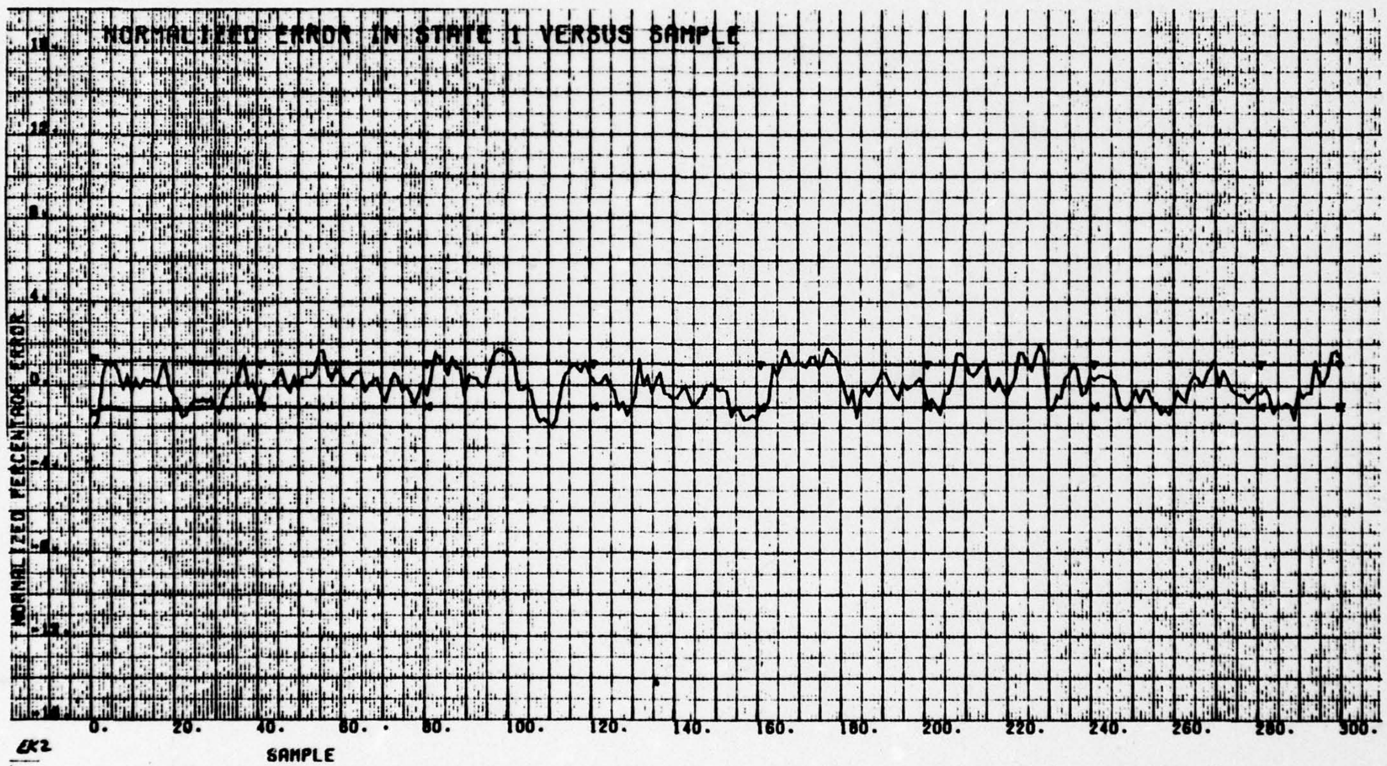

Figure 39. Extended Kalman Filter Estimate Error in State 1 Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.00025 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation —⊕—
Minus Estimate Standard Deviation —X—

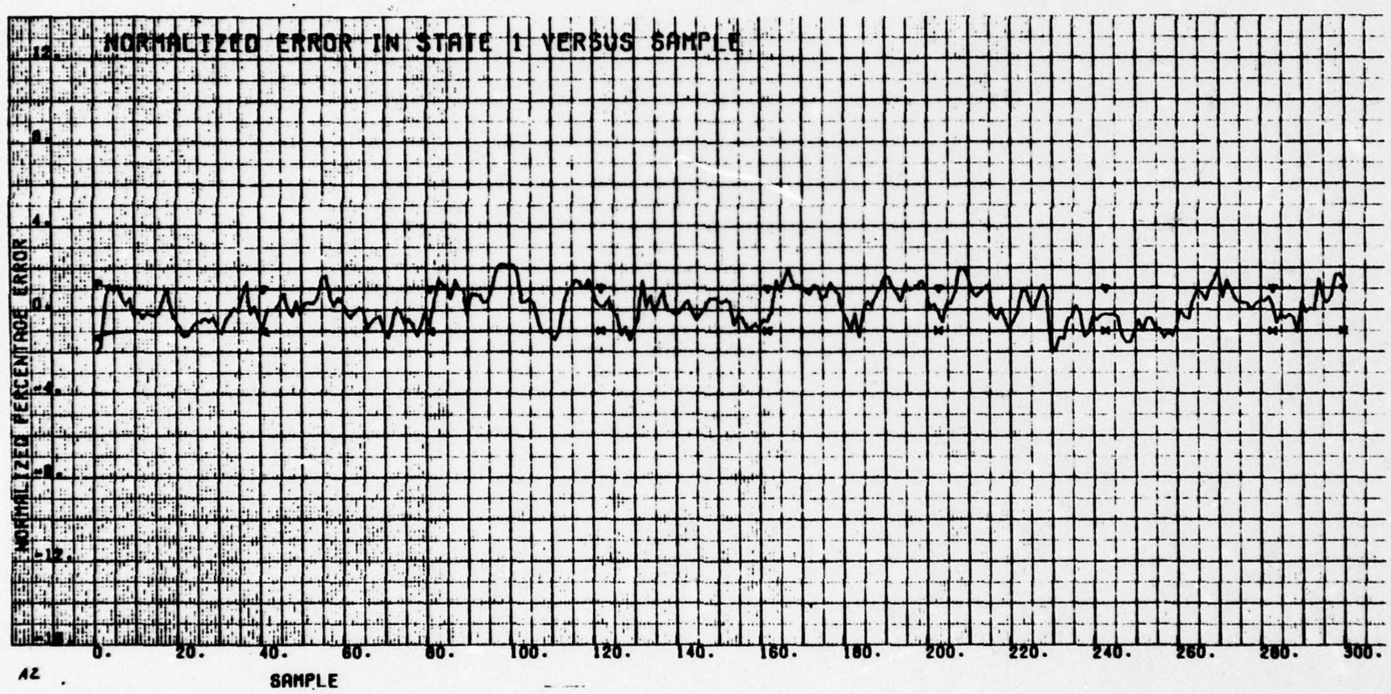

Figure 40. Adaptive Filter Estimate Error in State 1 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

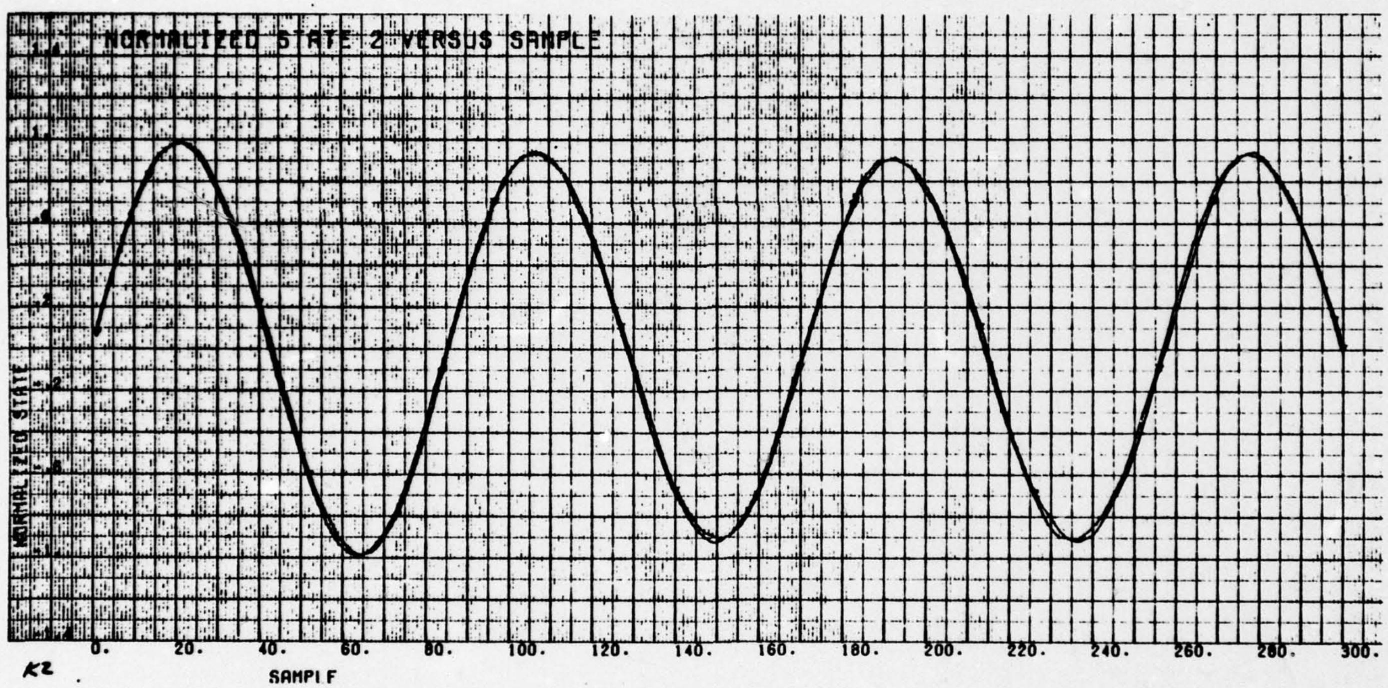

Figure 41. Kalman Filter Estimate of State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

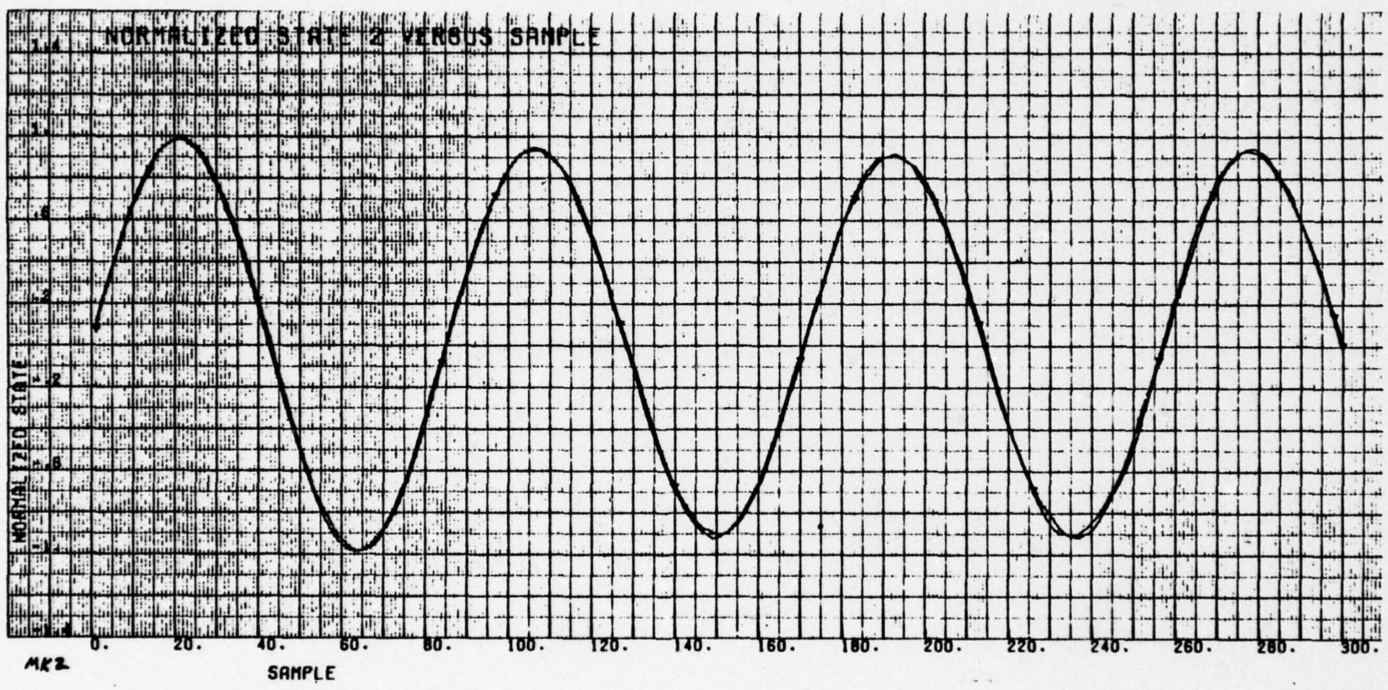

Figure 42. Modified Filter Estimate of State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

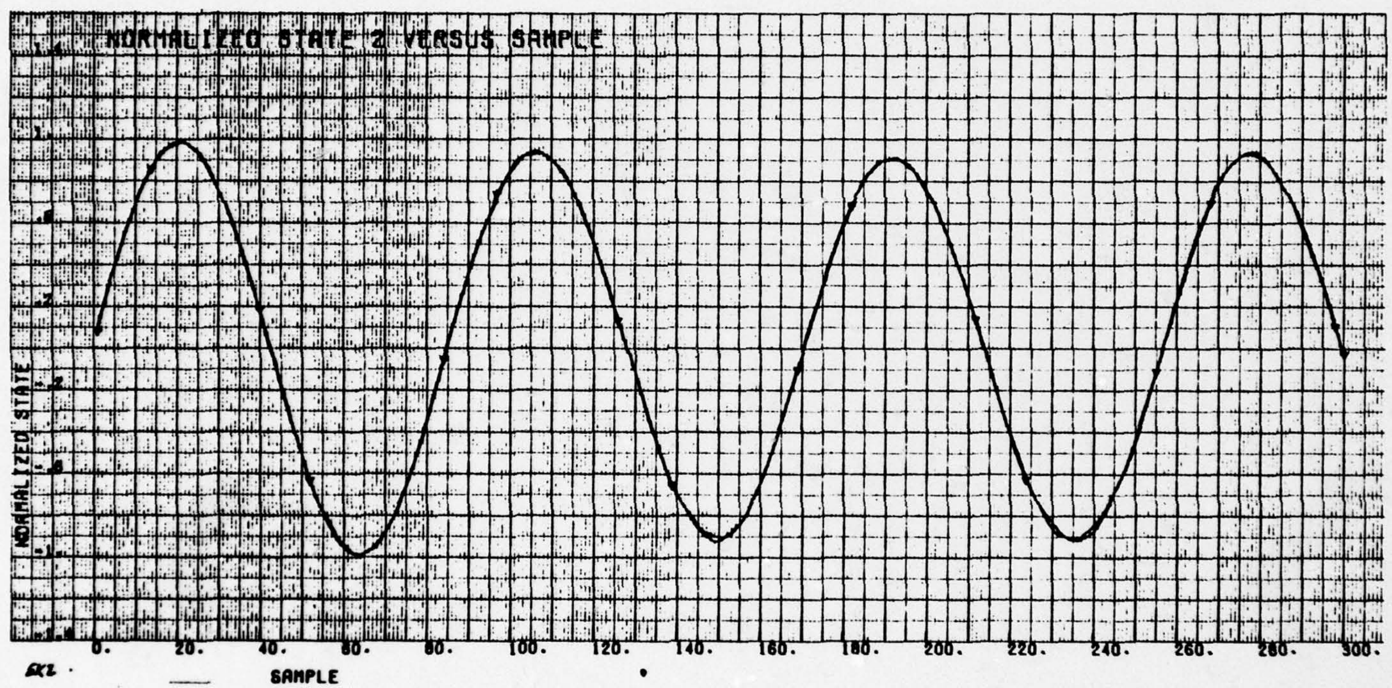

Figure 43. Extended Kalman Filter of State 2 Determined with Reduced Measurement Noise.

Initial State - 0.3 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.00025 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State

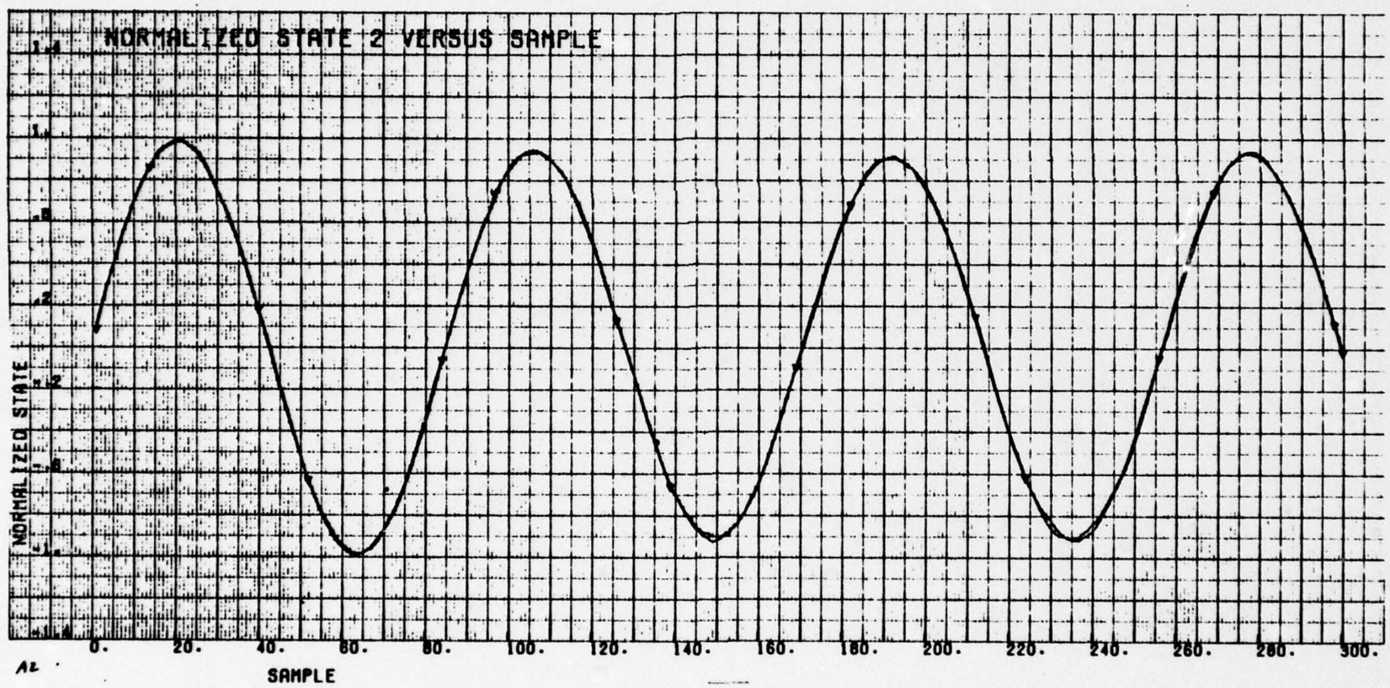

Figure 44. Adaptive Filter Estimate of State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

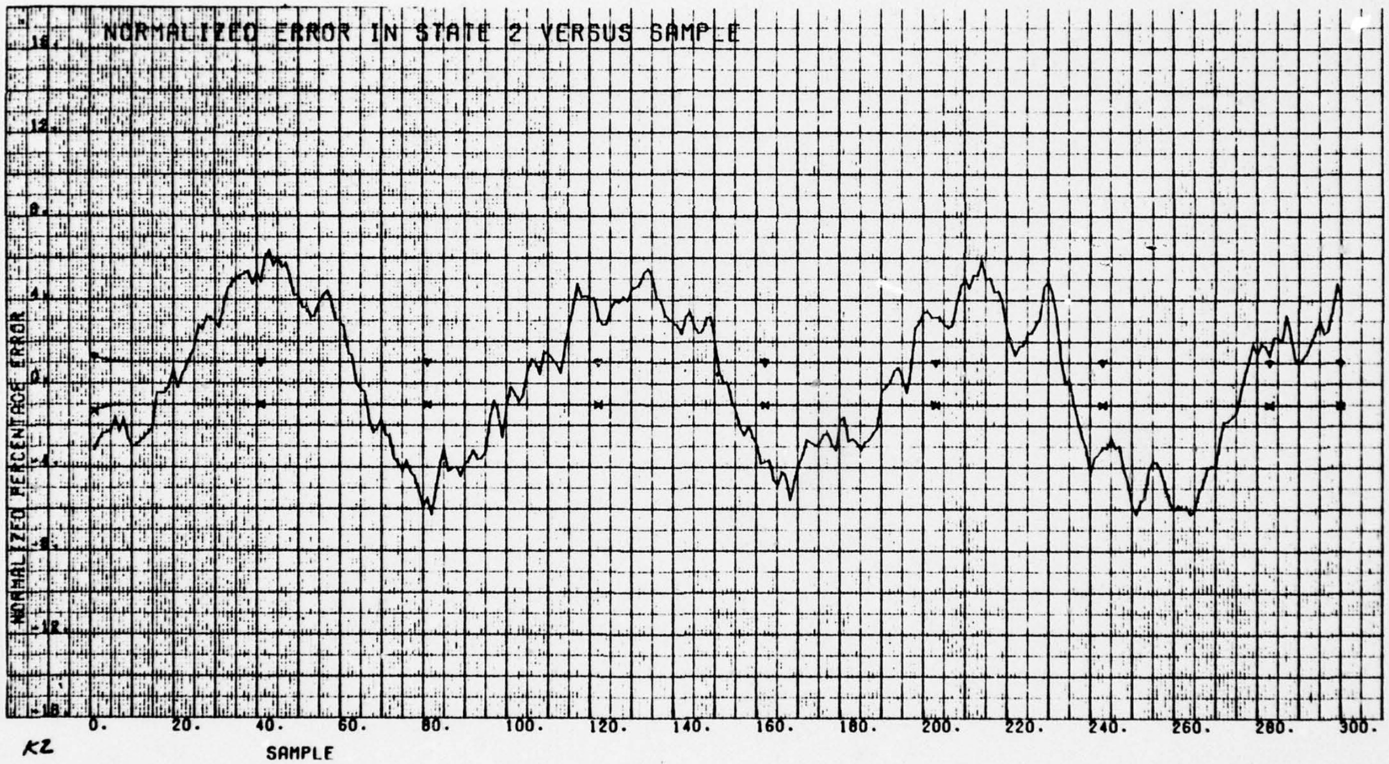

Figure 45. Kalman Filter Estimate Error in State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

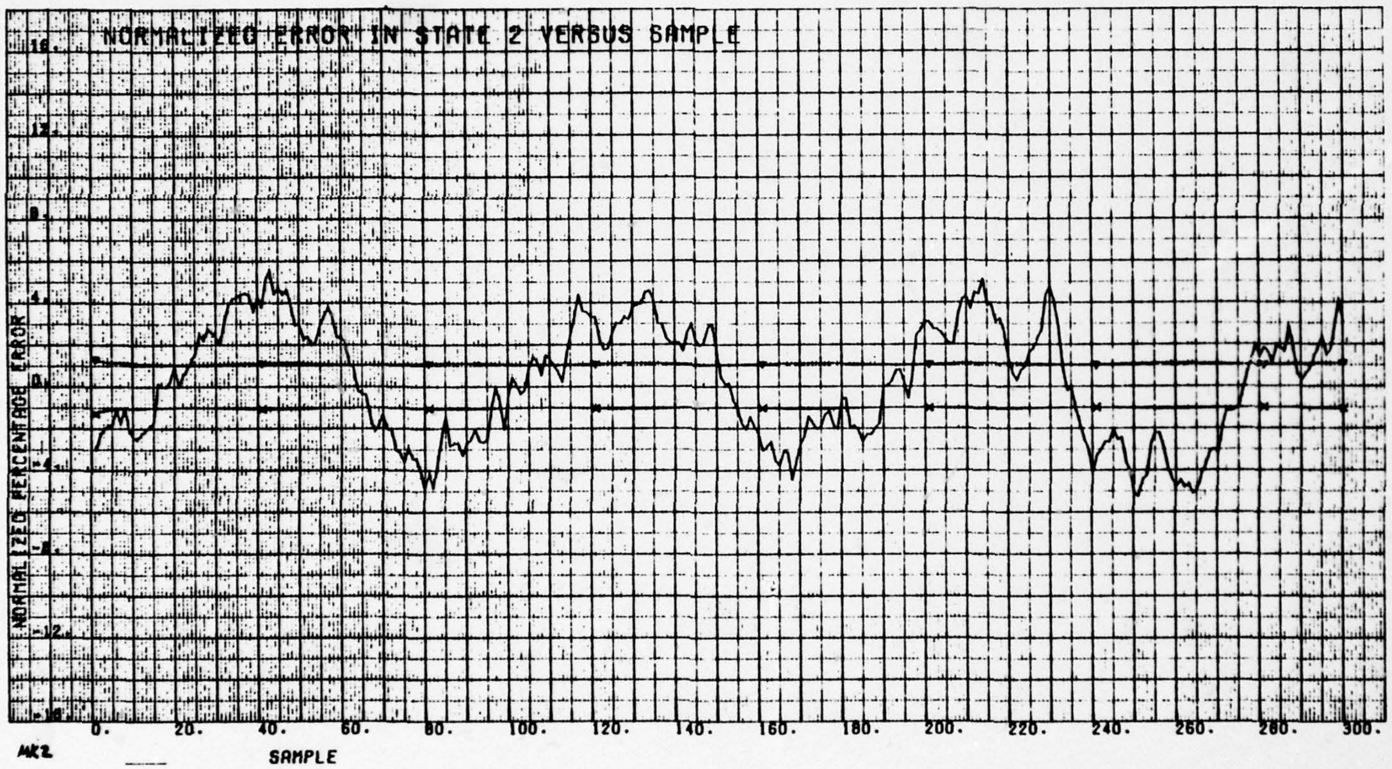

Figure 46. Modified Kalman Filter Estimate Error in State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec	Sample Rate - 0.1 sec
Initial Estimate - 0.0 rad/sec	Estimate Error ——
Normalizing Constant - 0.66 rad/sec	Plus Estimate Standard Deviation
Initial State Covariance - 0.0001 $(rad/sec)^2$	Minus Estimate Standard Deviation
System Noise Covariance - 0.00001 $(rad/sec)^2$	
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$	

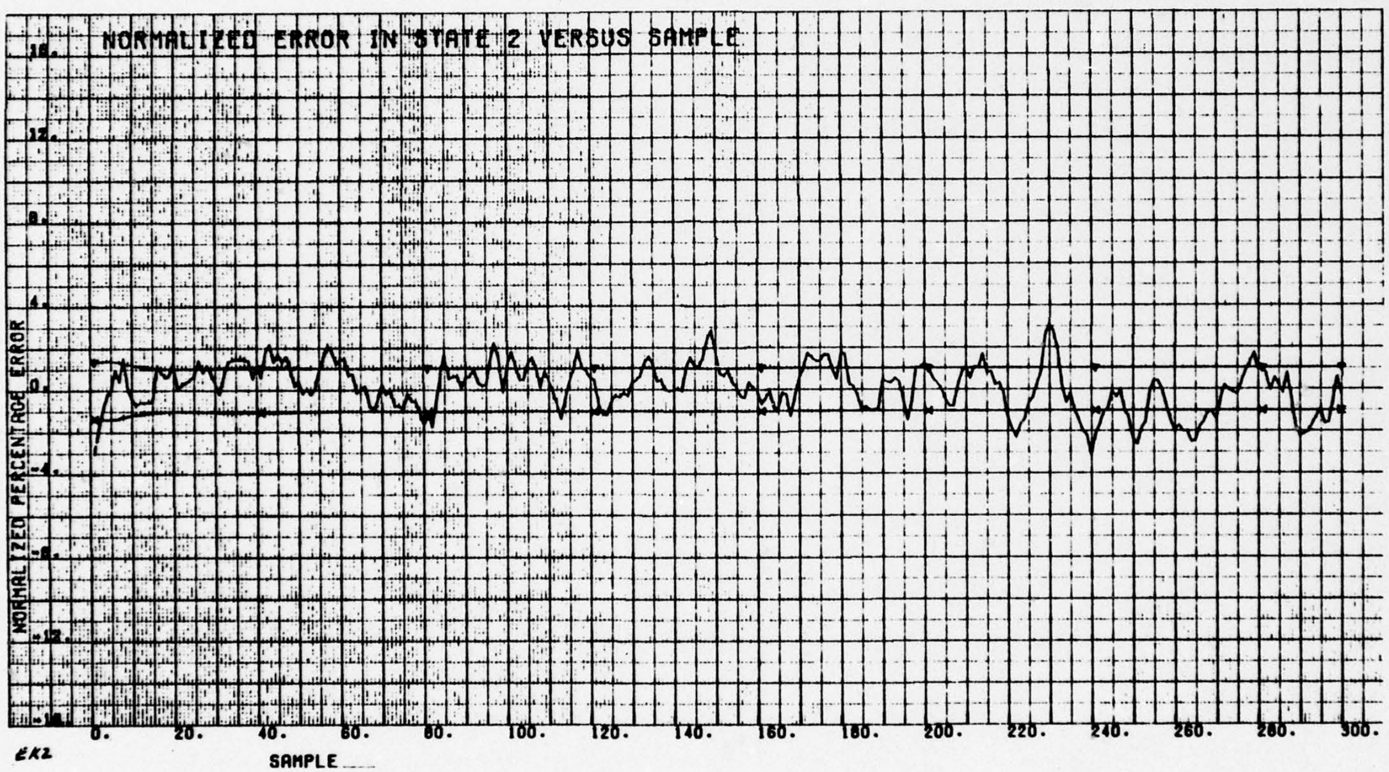

Figure 47. Extended Kalman Filter Estimate Error in State 2 Determined with Reduced Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

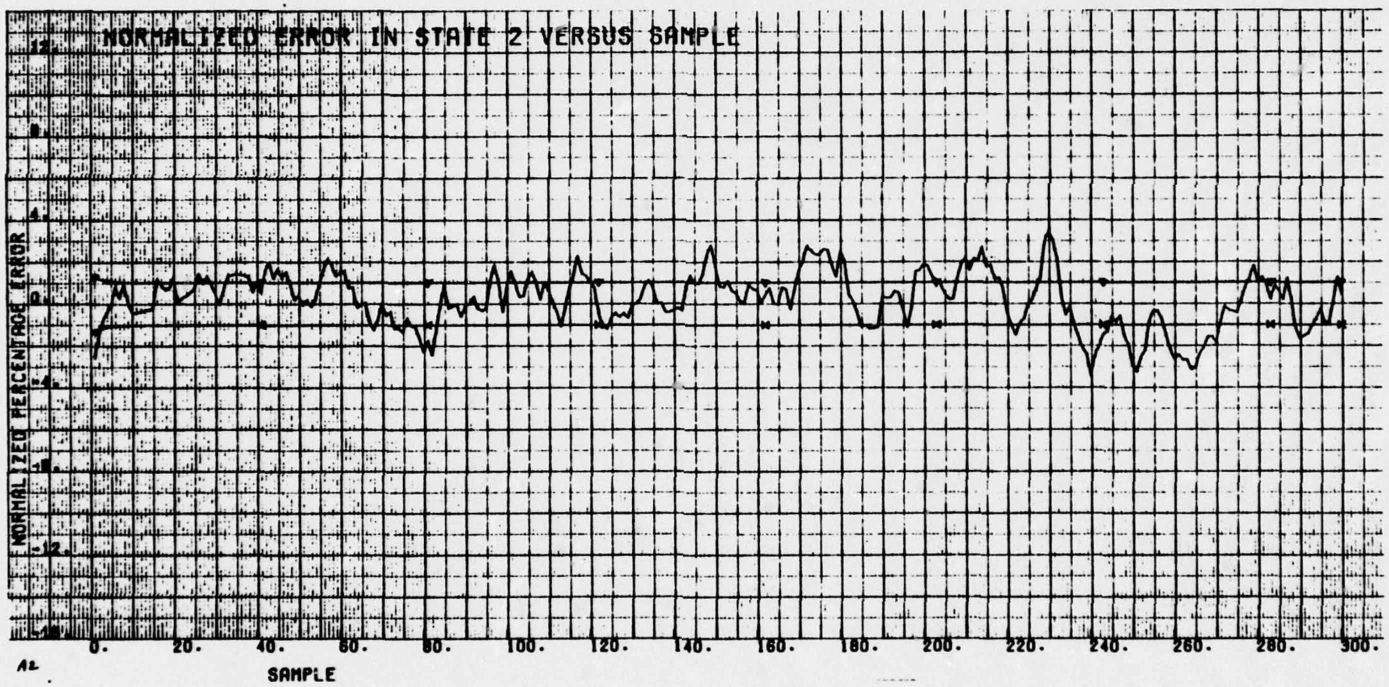

Figure 48. Adaptive Filter Estimate Error in State 2 Determined with Reduced Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.00025 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True Value ——
Estimated Value

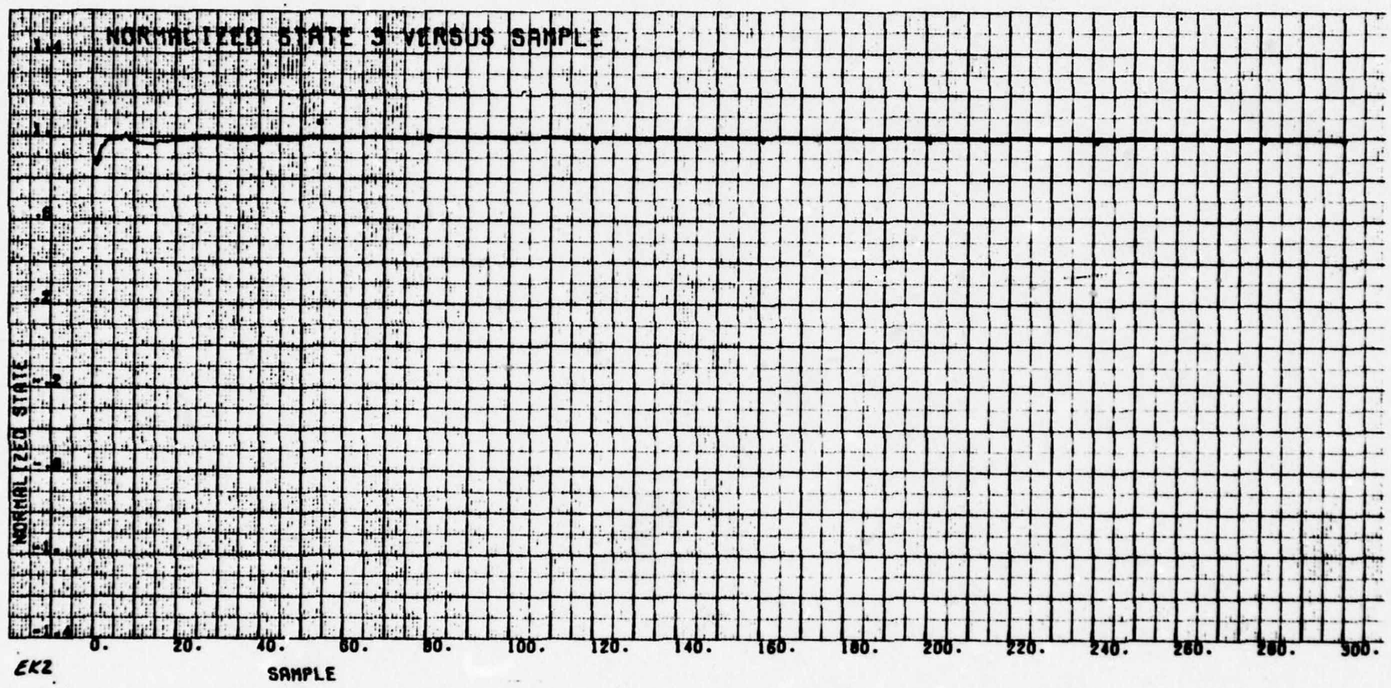

Figure 49. Extended Kalman Filter Estimate of the Uncertain Parameter in the System Determined with Reduced Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.00025 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True Value ——
Estimated Value —▽—

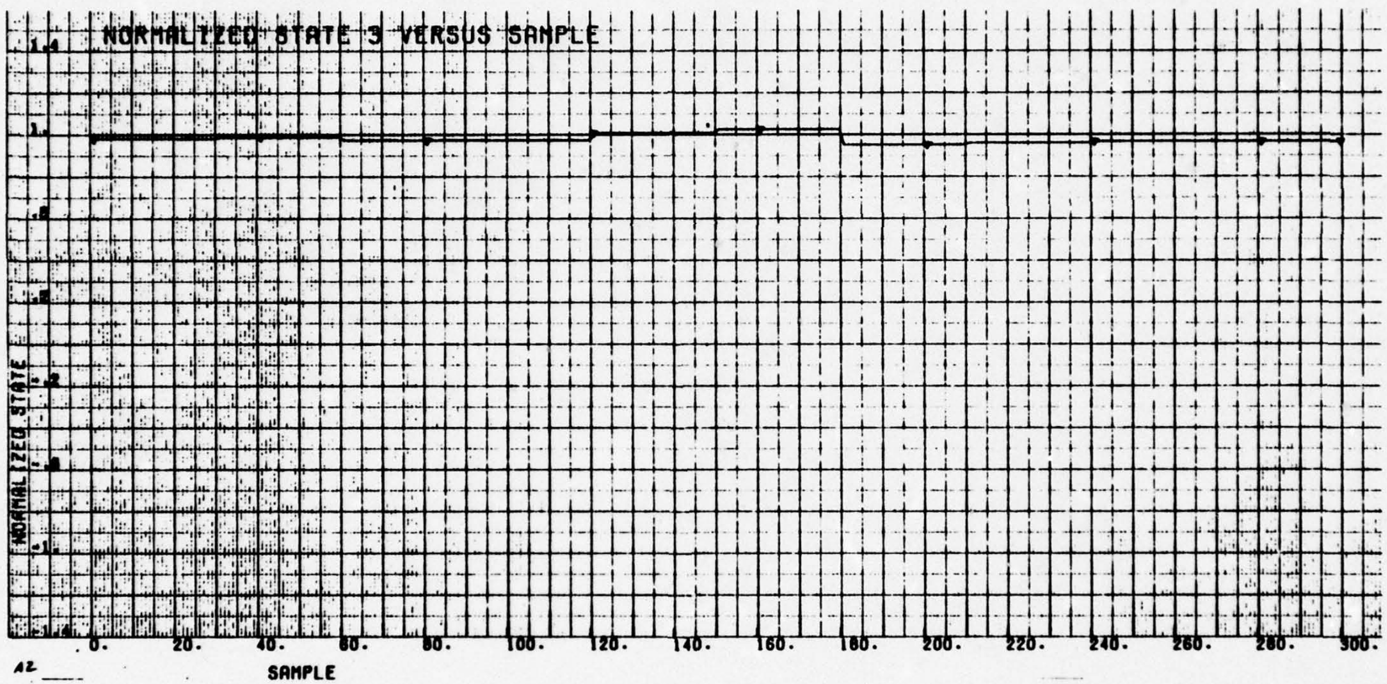

Figure 50. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics Determined with Reduced Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Value - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$
Sample Rate - 0.1 sec

Estimate Error ——
Plus Estimate Standard Deviation —▽—
Minus Estimate Standard Deviation —X—

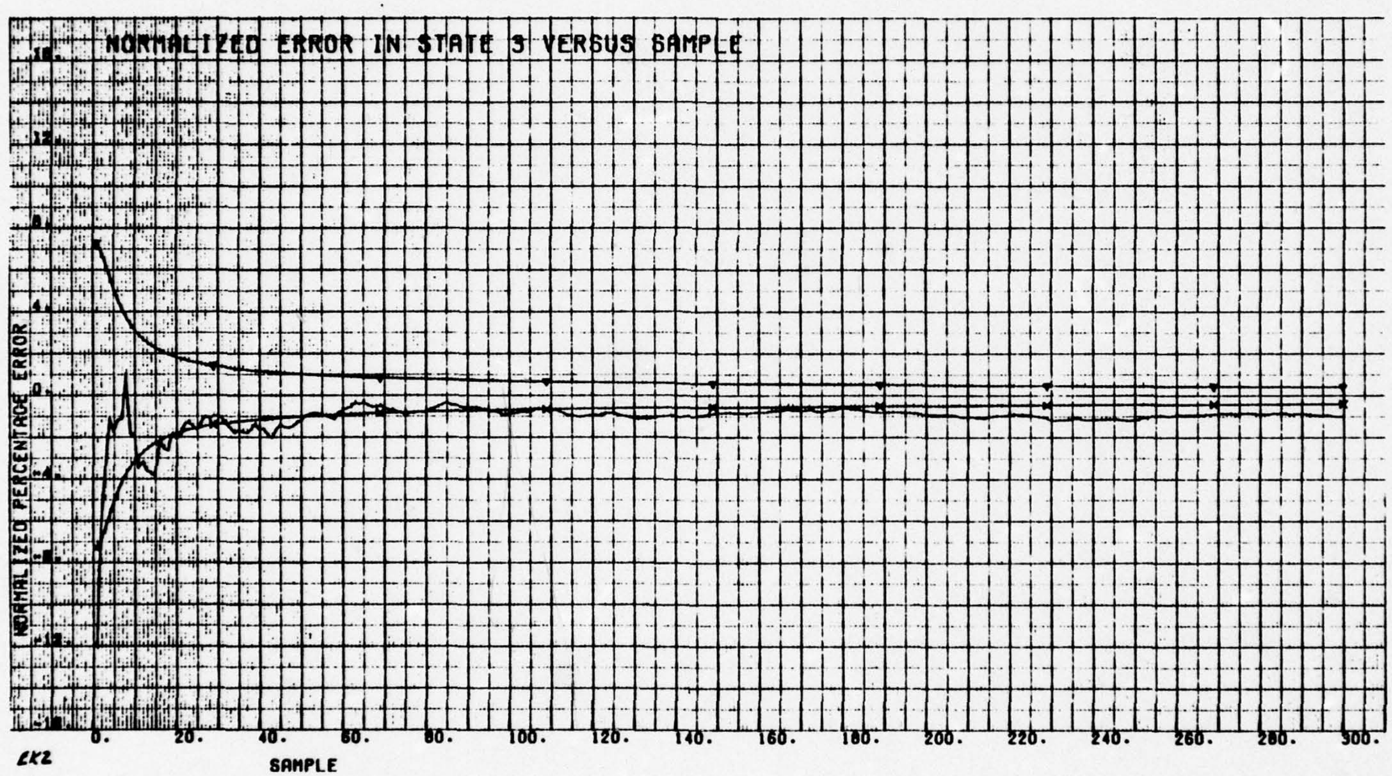

Figure 51. Extended Kalman Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with Reduced Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Value - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.00025 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation ——

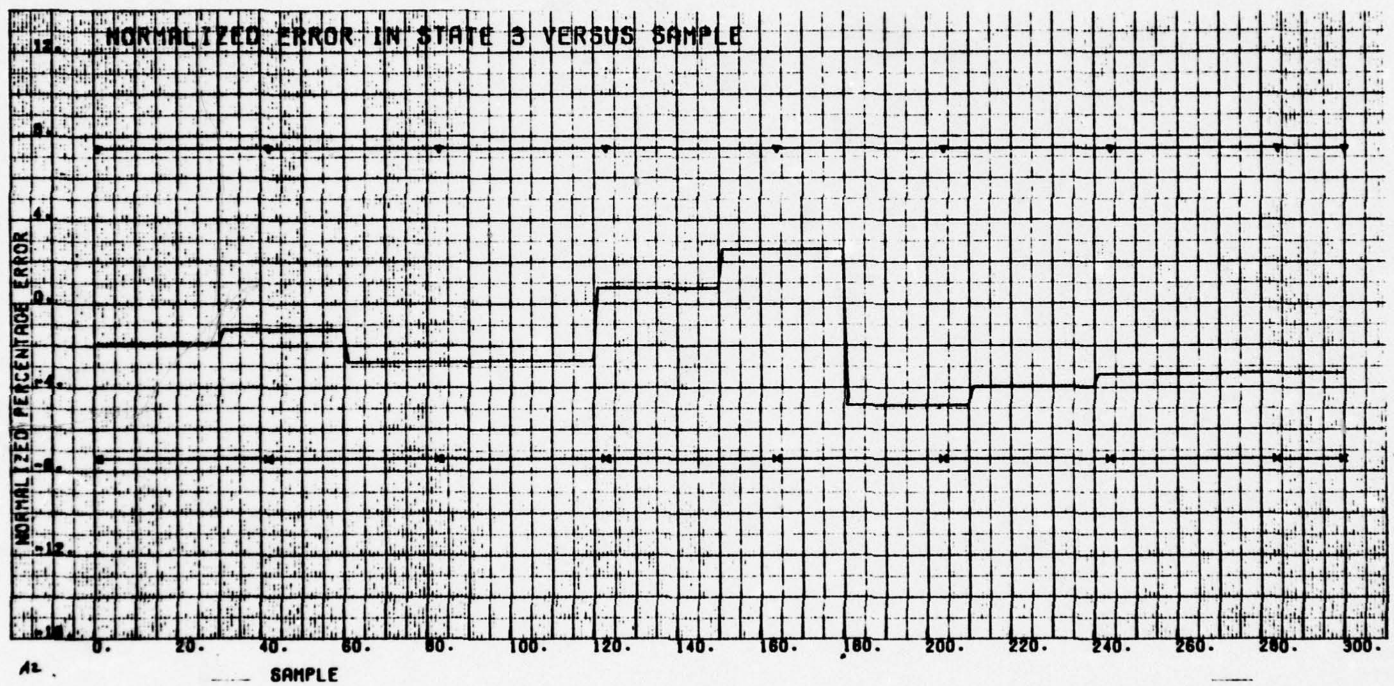

Figure 52. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with Reduced Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
True State ——
Estimated State —▽—

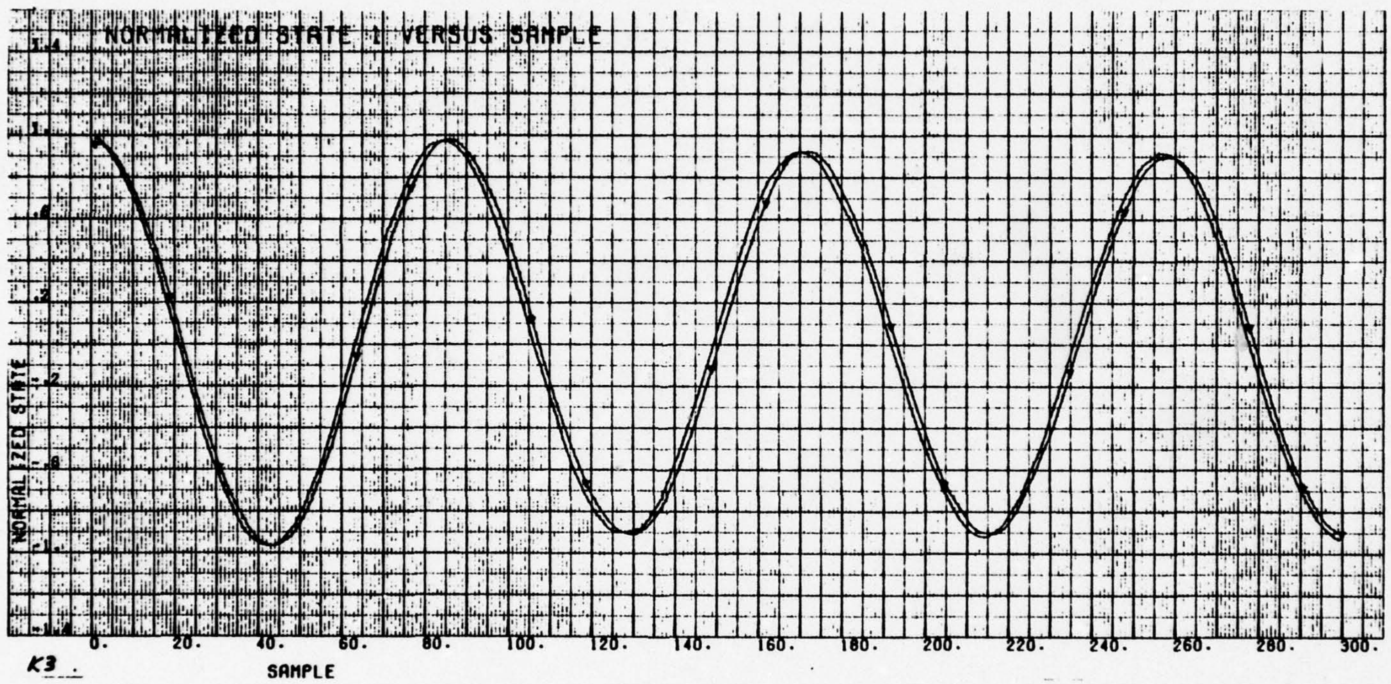

Figure 53. Kalman Filter Estimate of State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

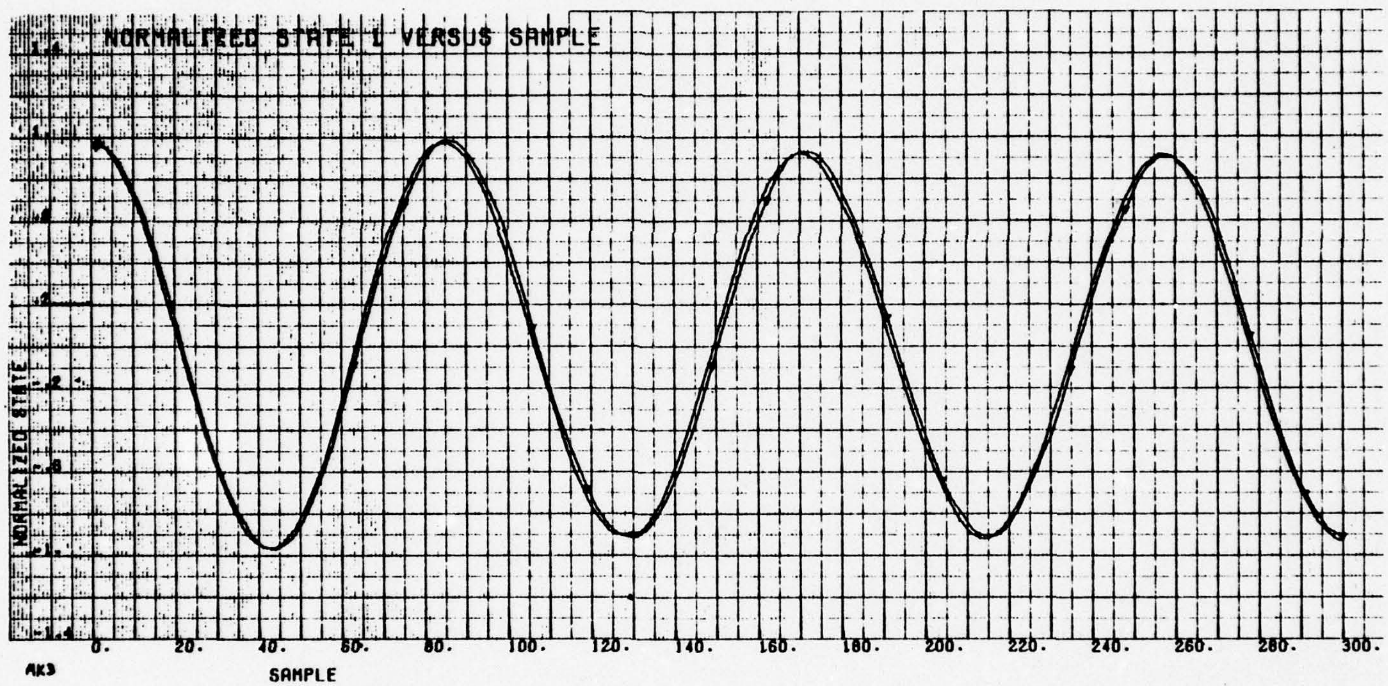

Figure 54. Modified Kalman Filter Estimate of State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

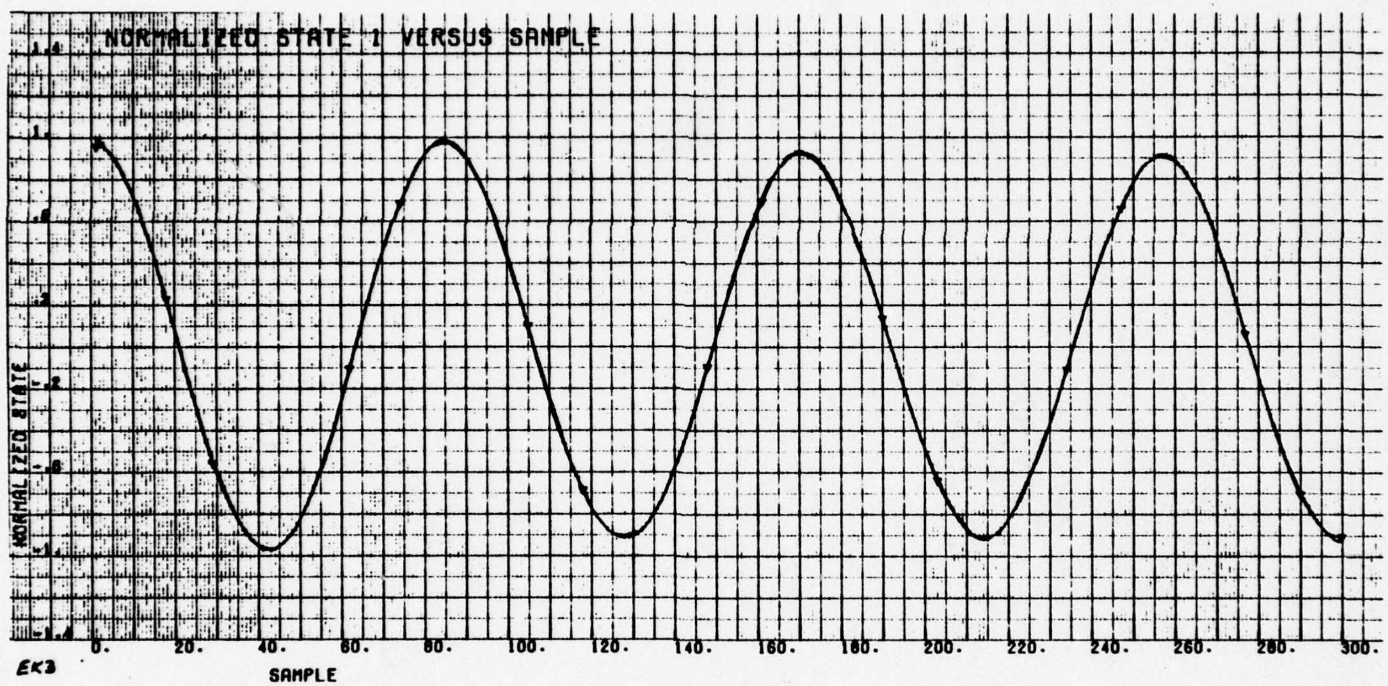

Figure 55. Extended Kalman Filter Estimate of State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimated State —▽—

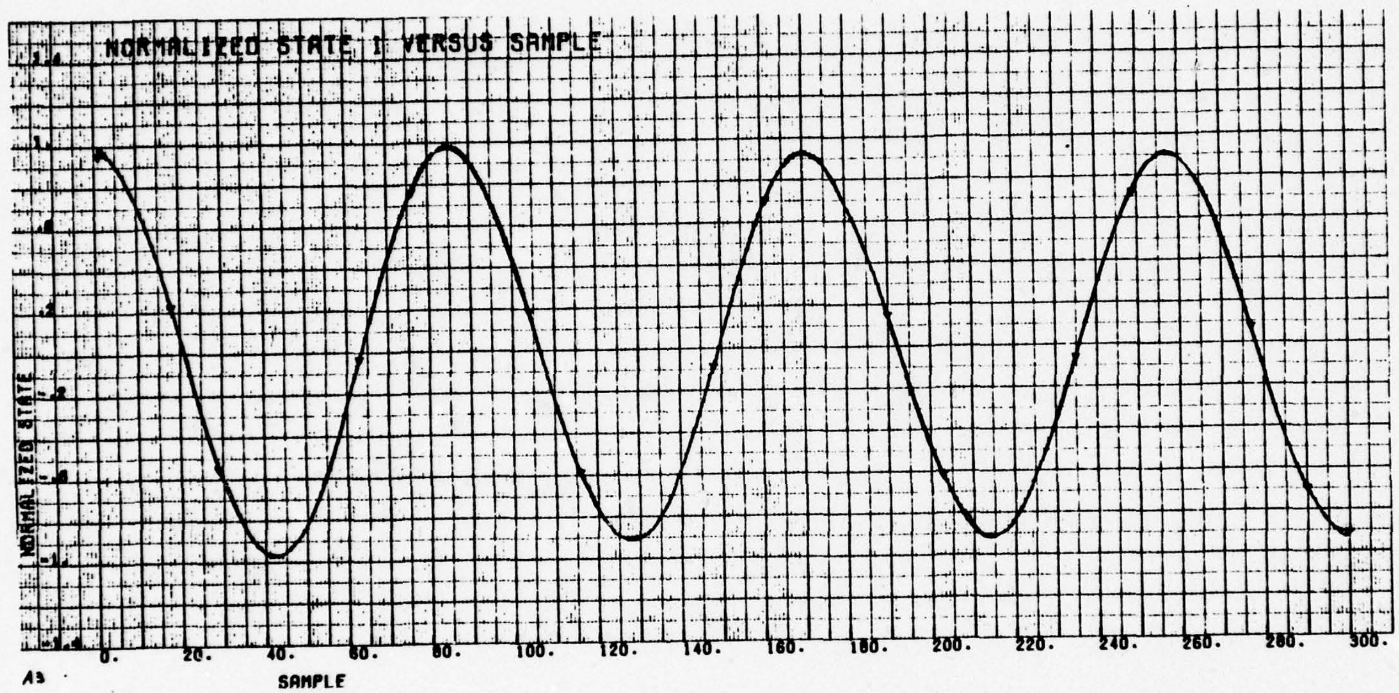

Figure 56. Adaptive Filter Estimate of State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

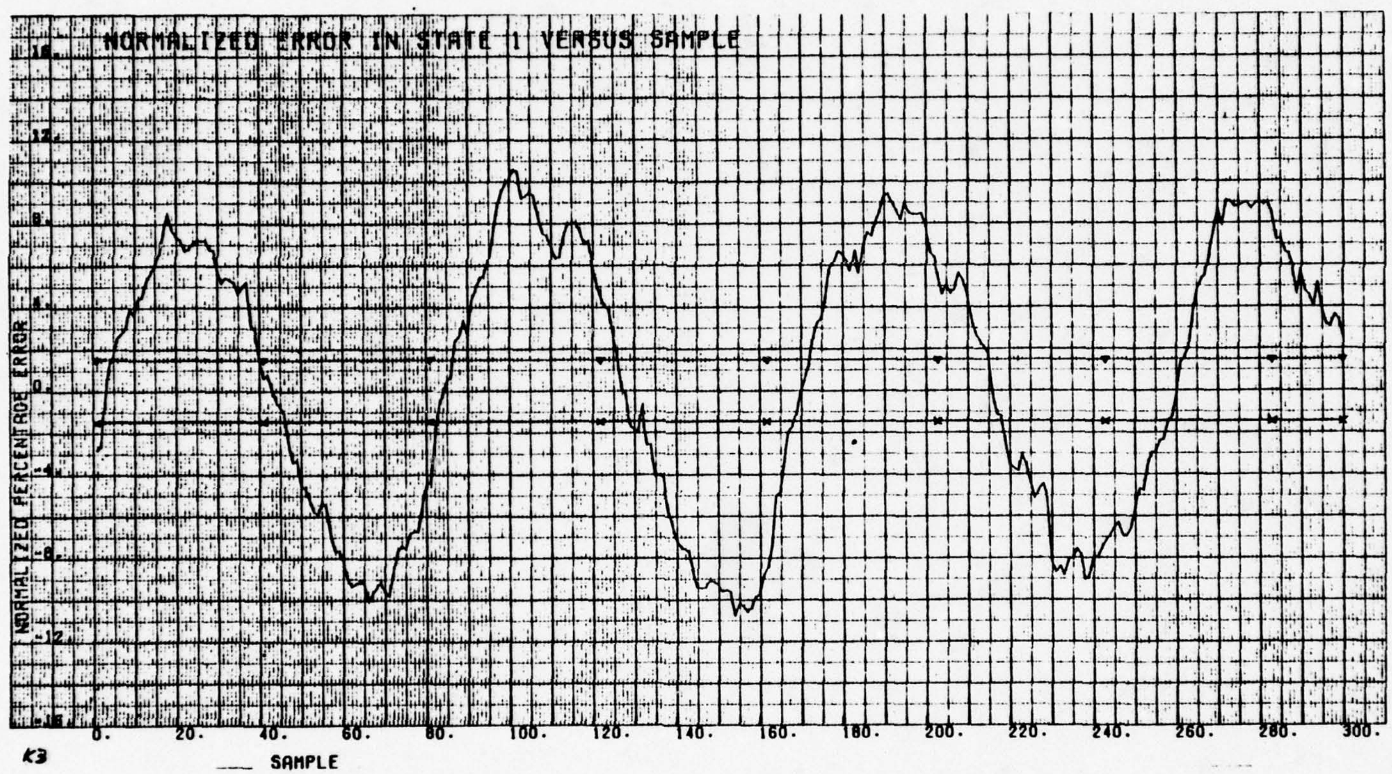

Figure 57. Kalman Filter Estimate Error in State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation —▽—
Minus Estimate Standard Deviation —X—

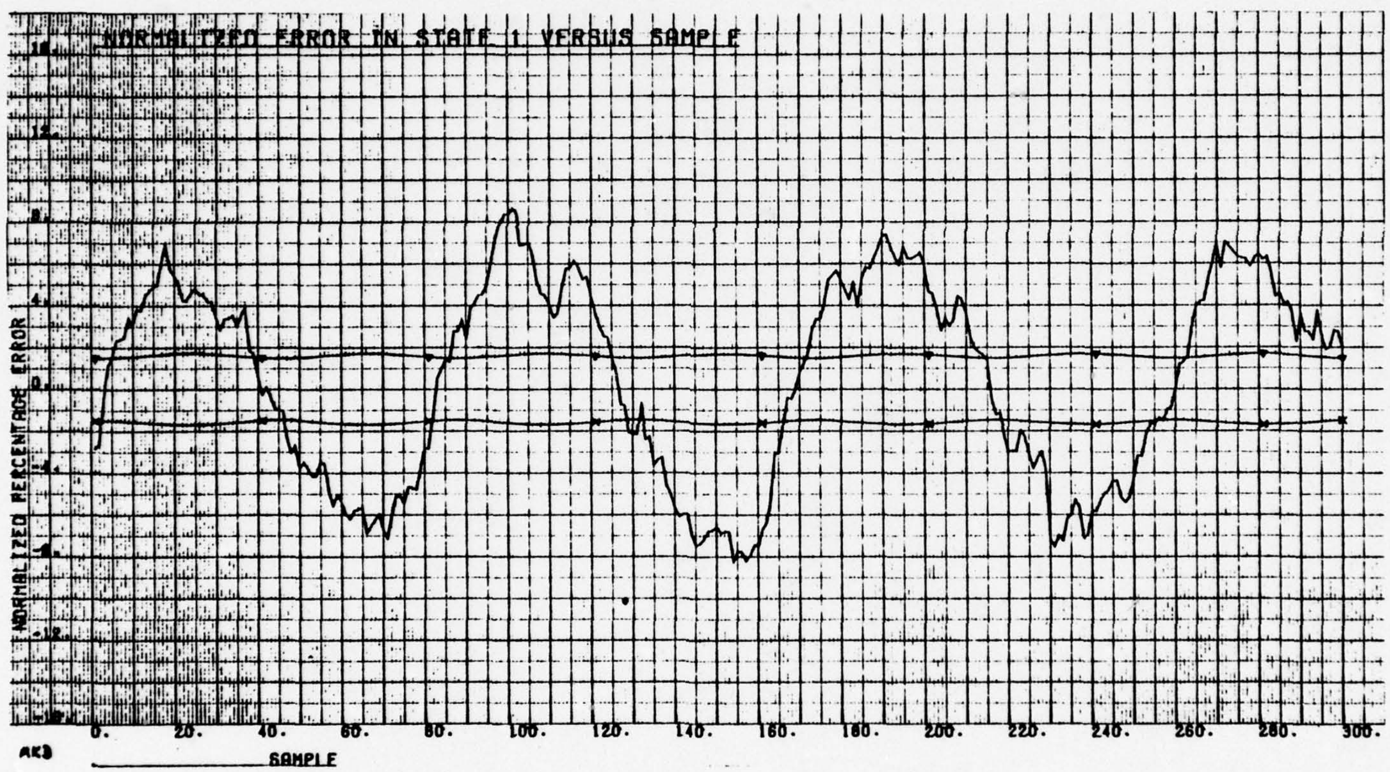

Figure 58. Modified Kalman Filter Estimate Error in State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

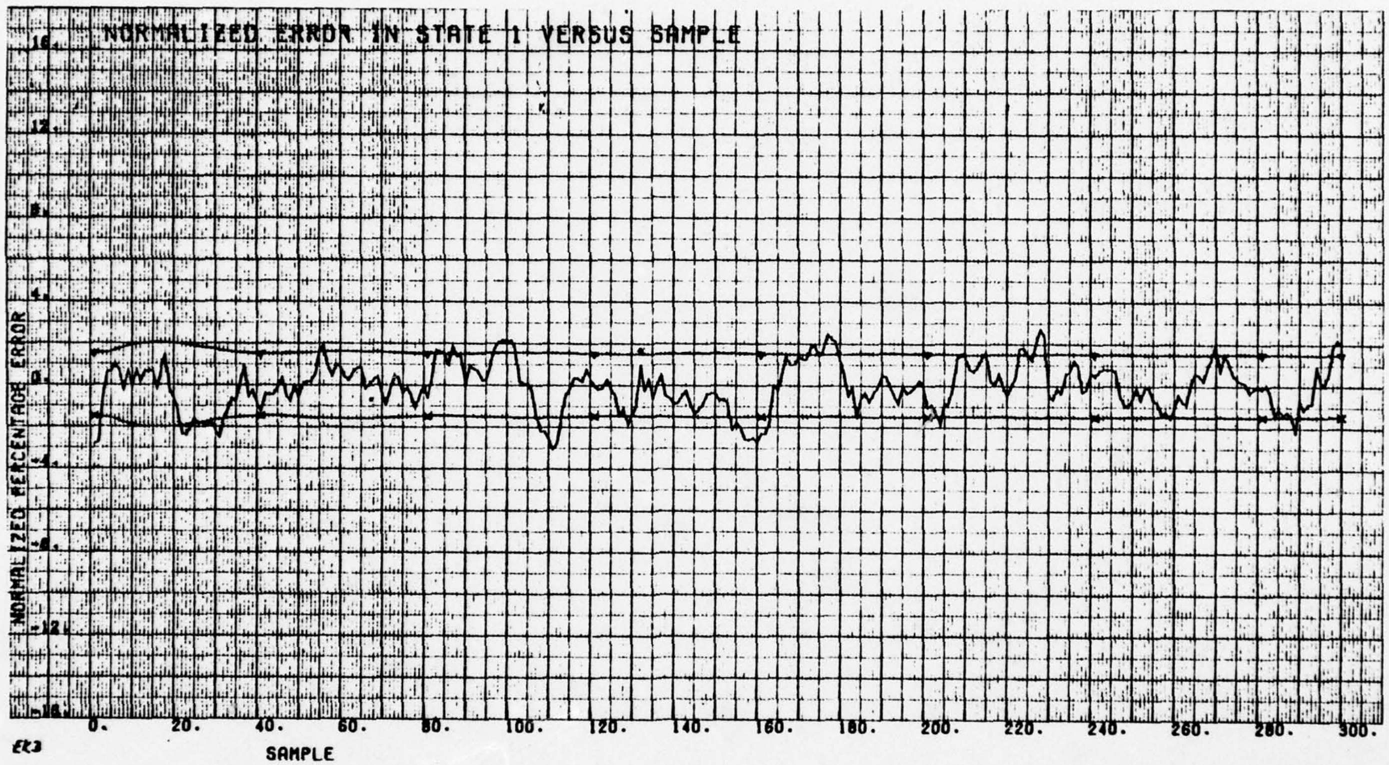

Figure 59. Extended Kalman Filter Estimate Error in State 1 Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation —∇—
Minus Estimate Standard Deviation —X—

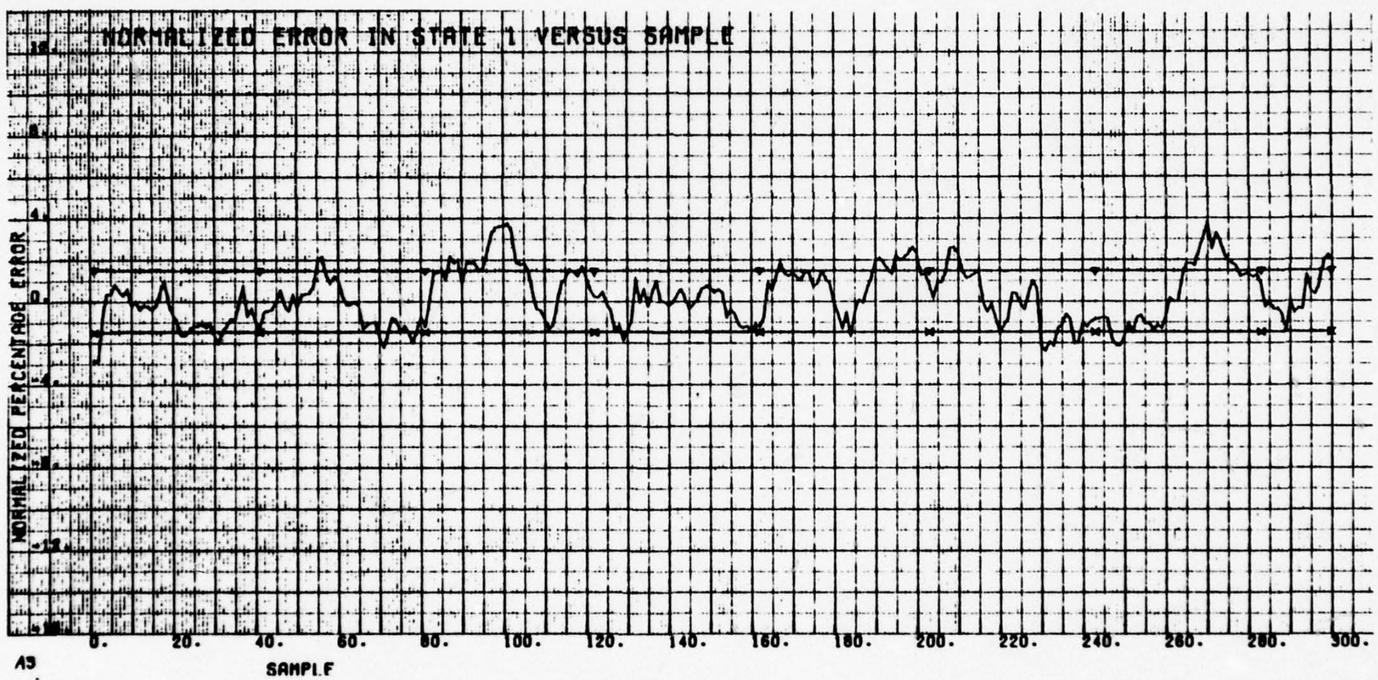

Figure 60. Adaptive Filter Estimate Error in State 1 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State —▽—

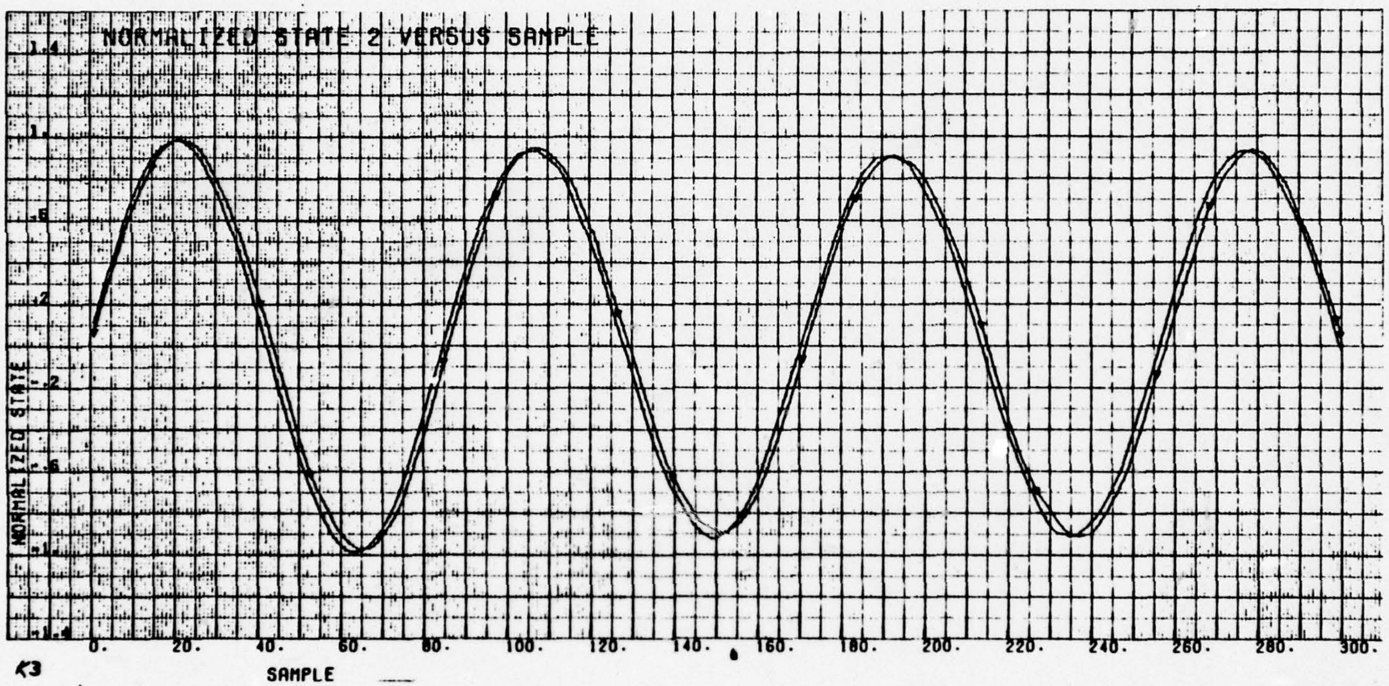

Figure 61. Kalman Filter Estimate of State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State —▽—

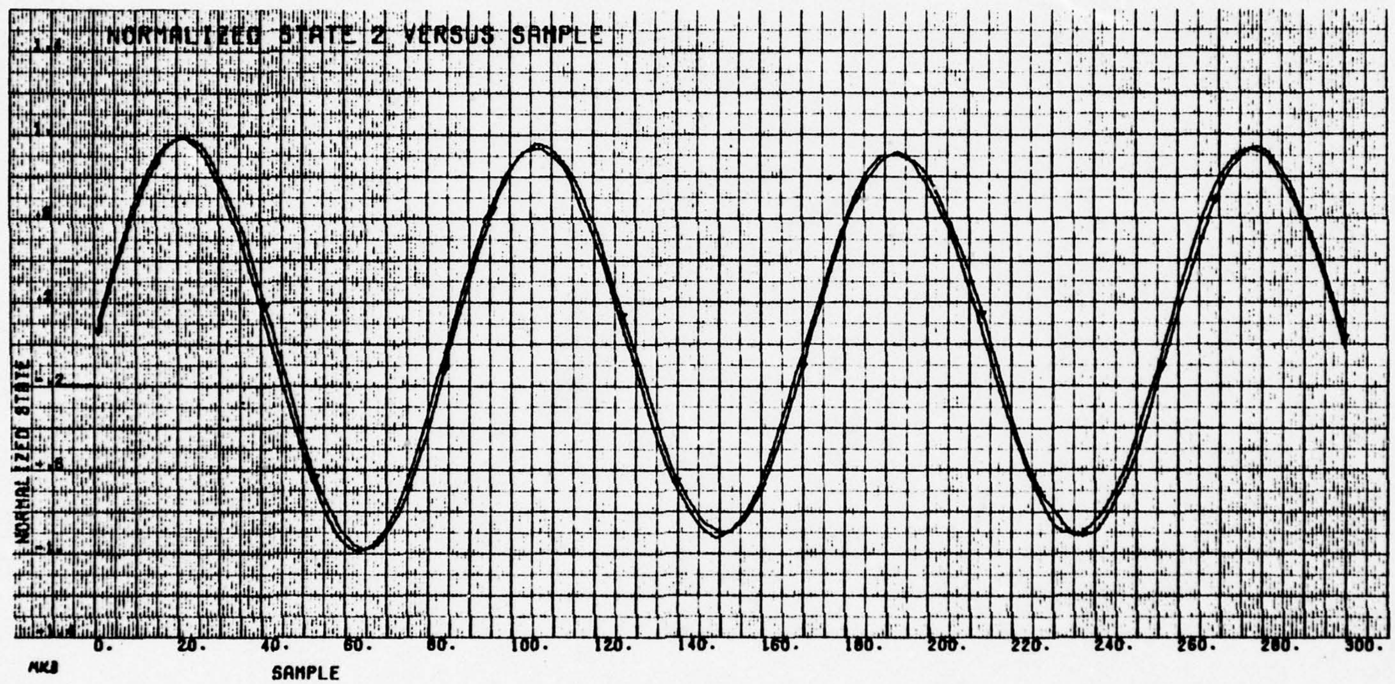

Figure 62. Modified Kalman Filter Estimate of State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True State ——
Estimated State

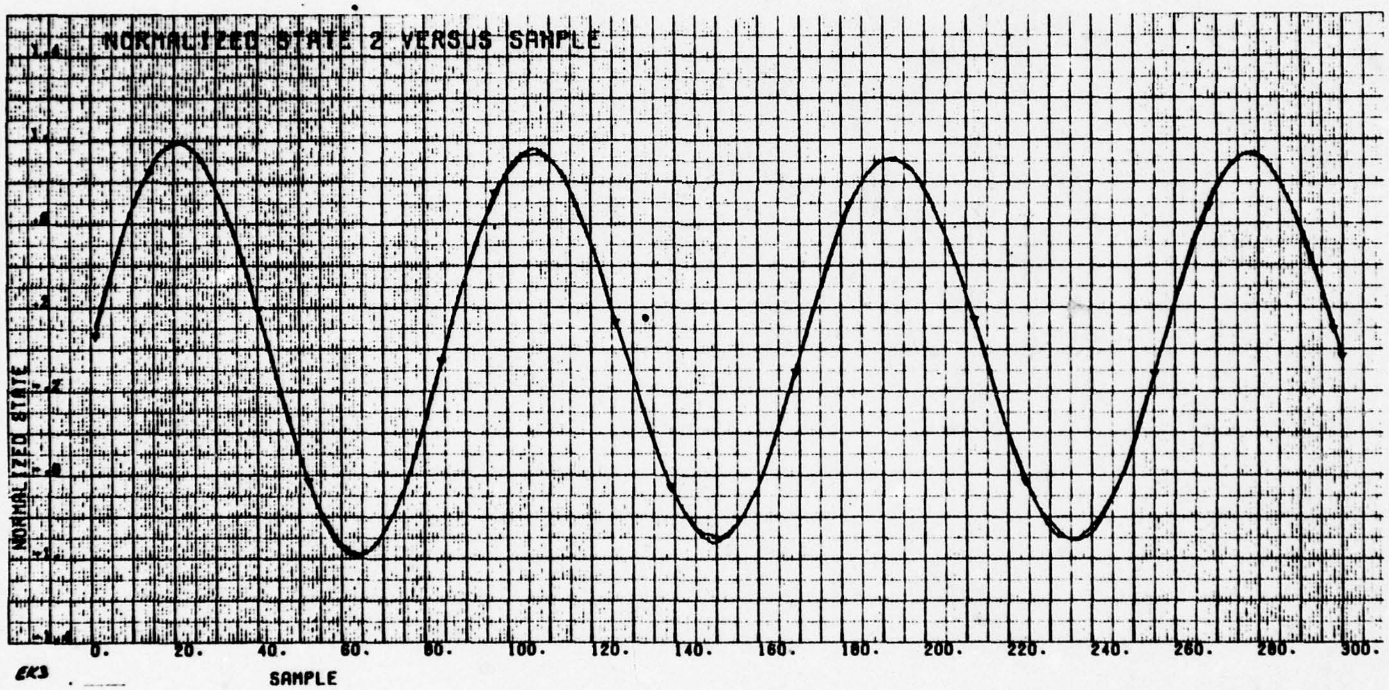

Figure 63. Extended Kalman Filter Estimate of State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True State ——
Estimate State —⊽—

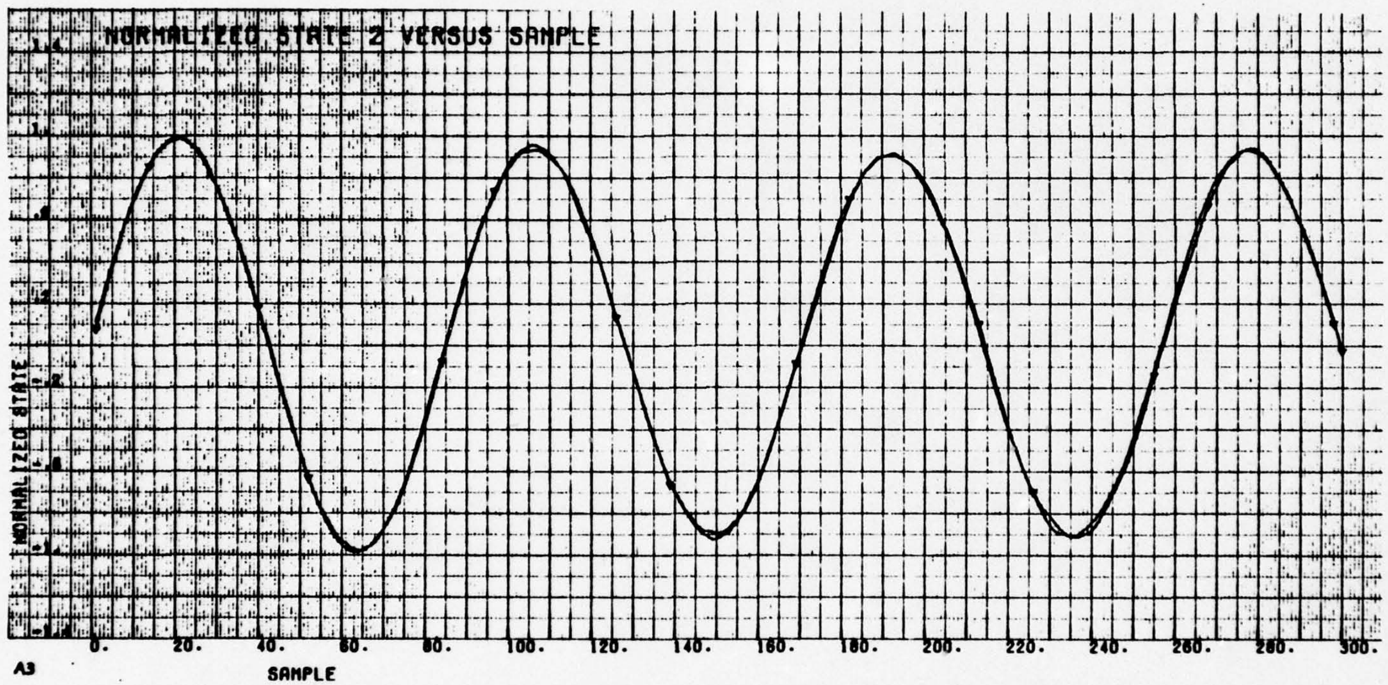

Figure 64. Adaptive Filter Estimate of State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation —⊽—
Minus Estimate Standard Deviation —*—

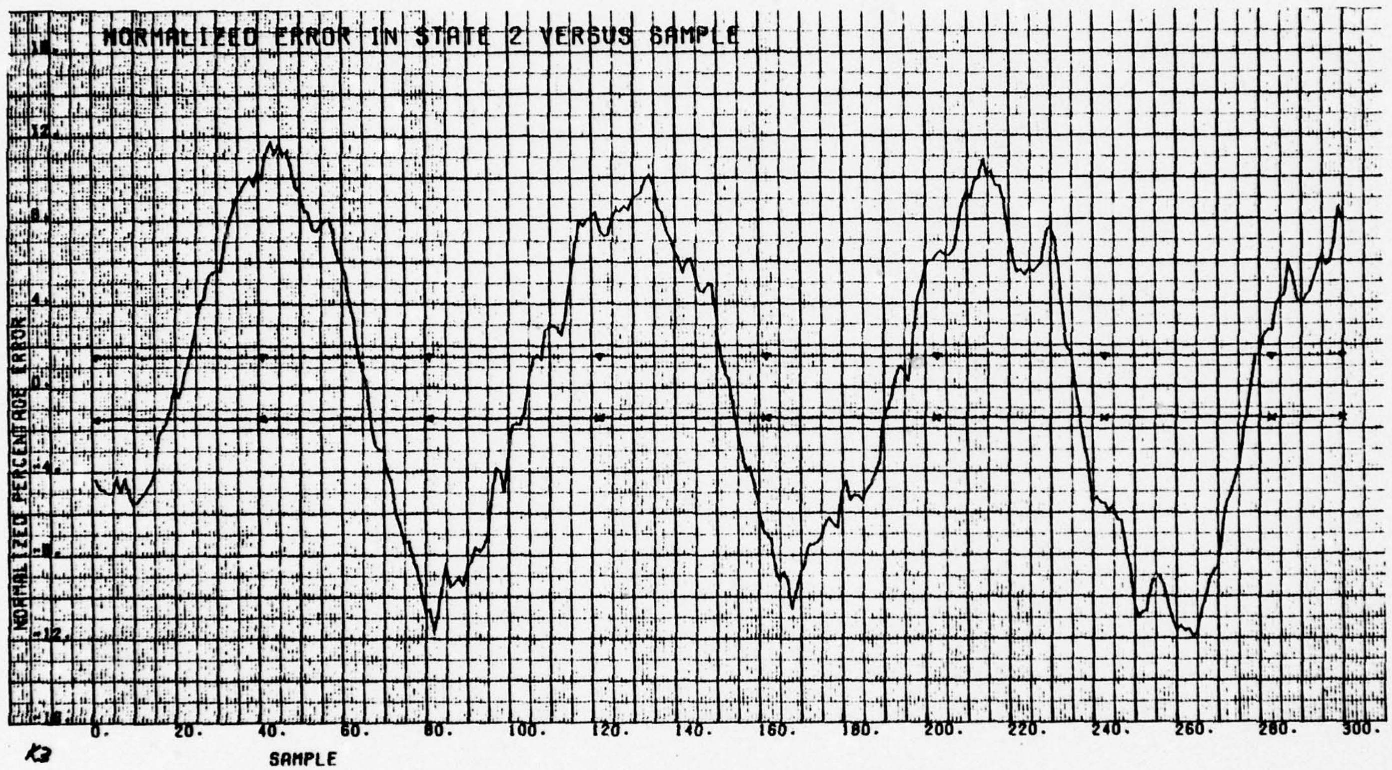

Figure 65. Kalman Filter Estimate Error in State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

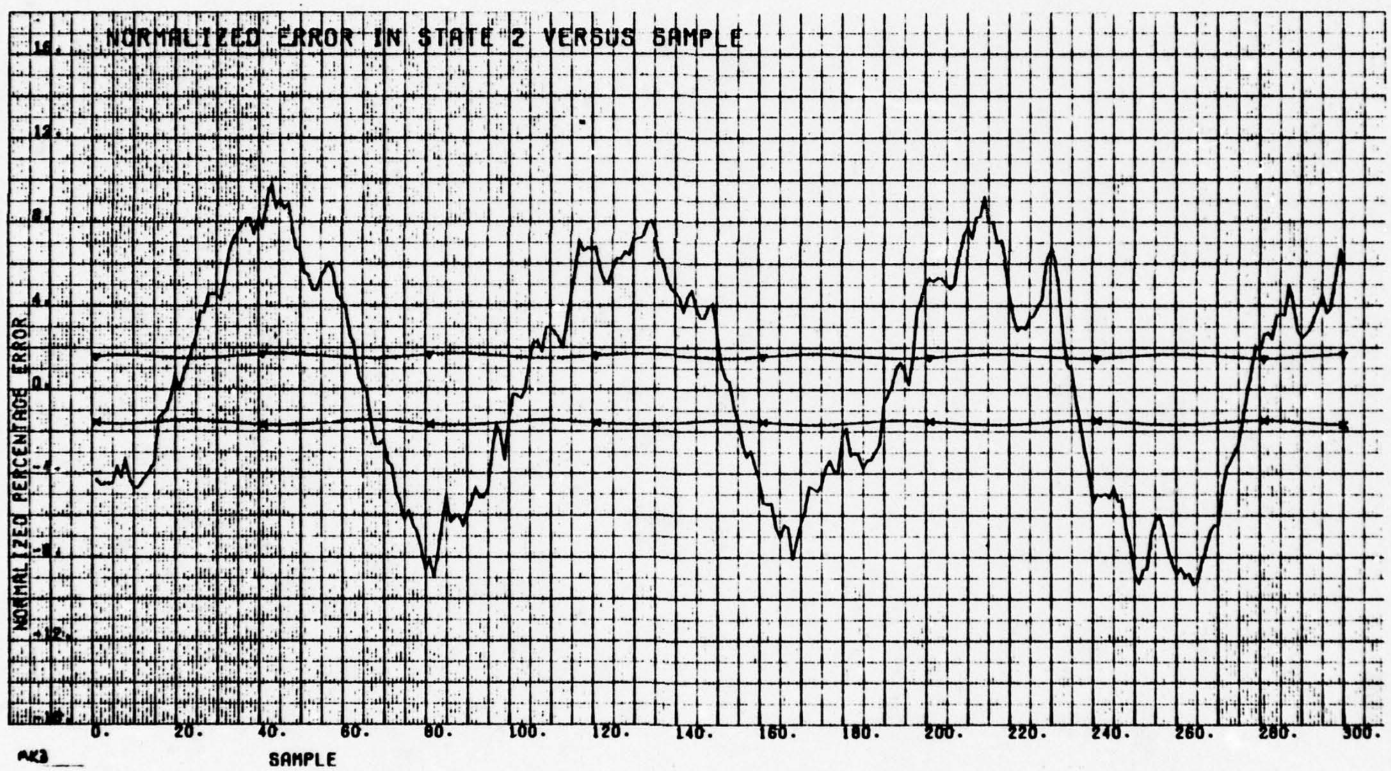

Figure 66. Modified Filter Estimate Error in State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

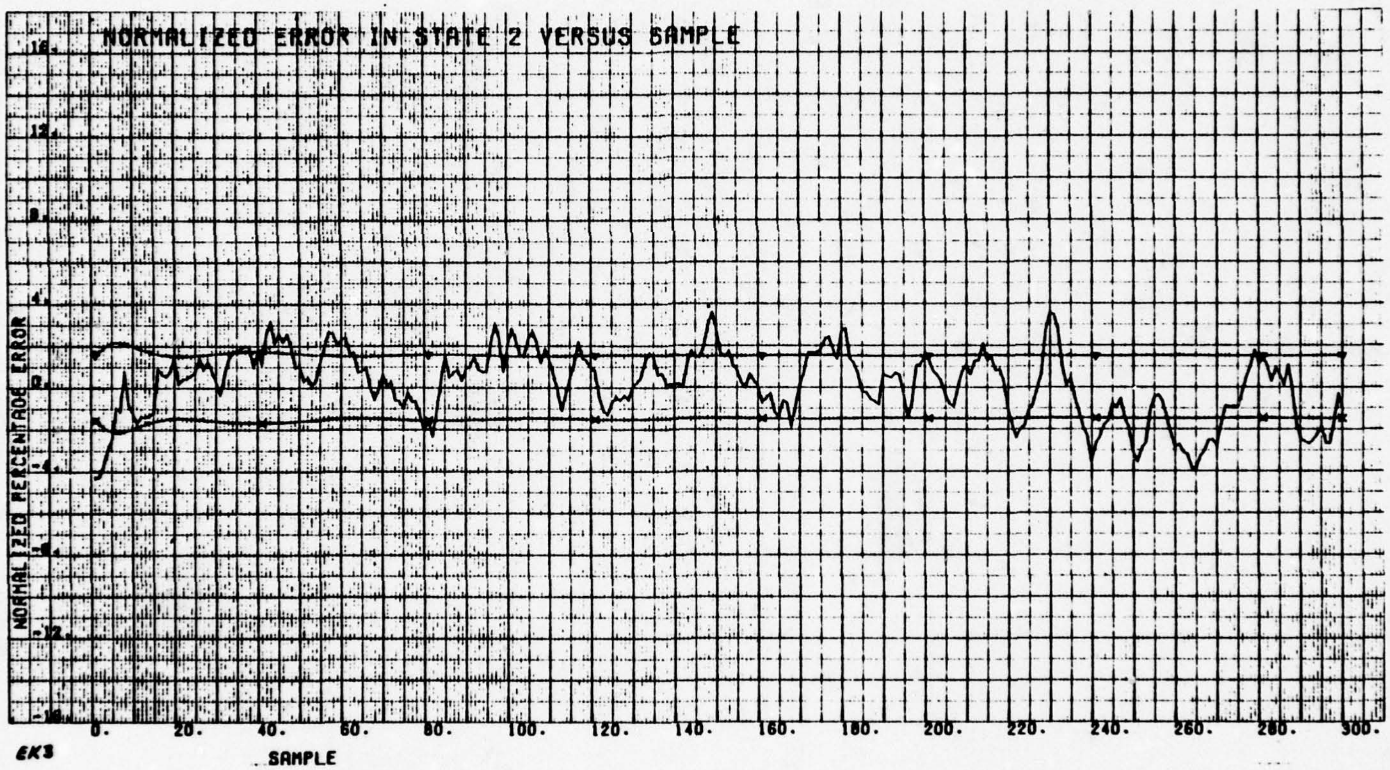

Figure 67. Extended Kalman Filter Estimate Error in State 2 Determined with Increased Measurement Noise.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

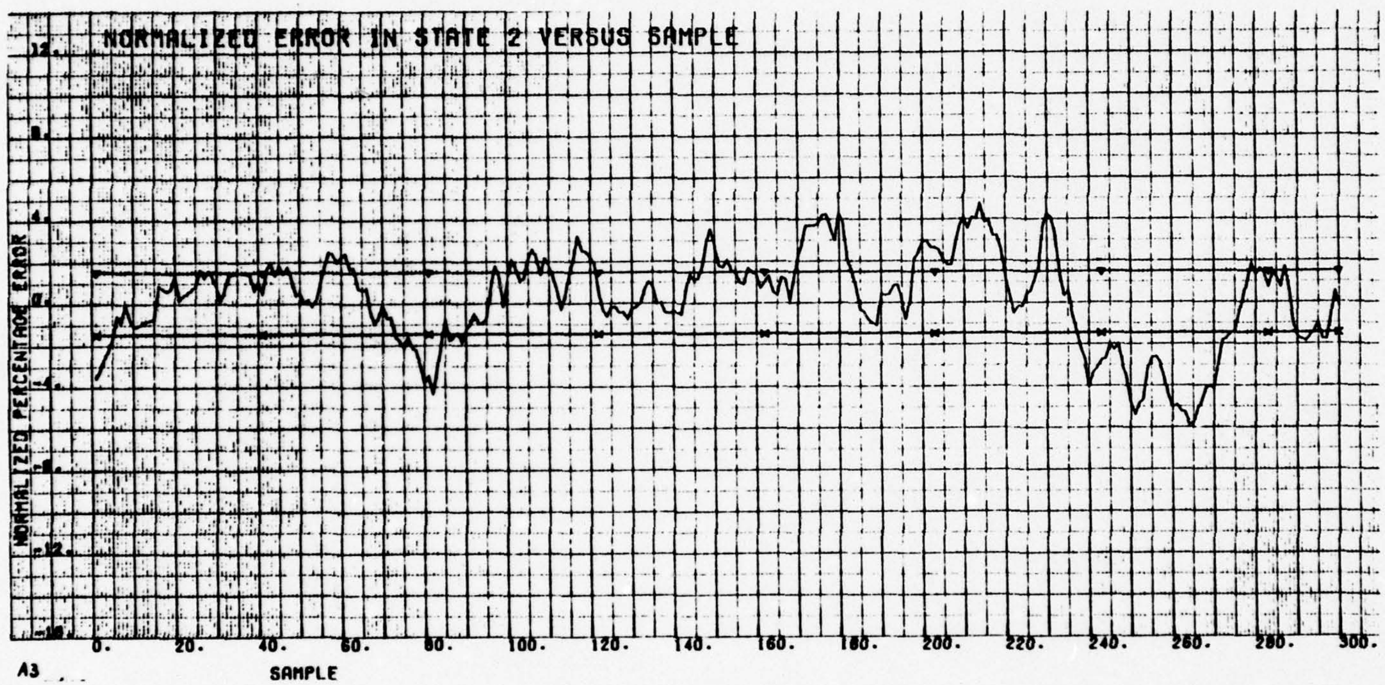

Figure 68. Adaptive Filter Estimate Error in State 2 Determined with Increased Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
True Value ——
Estimated Value

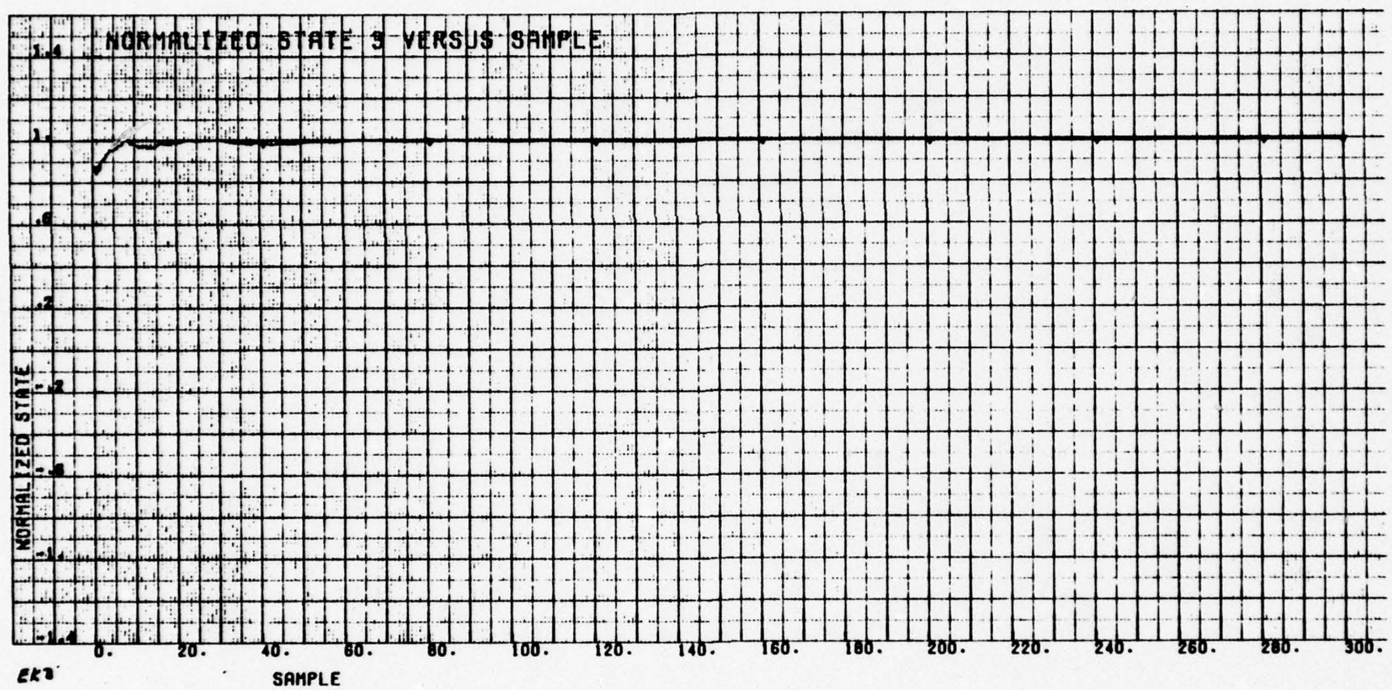

Figure 69. Extended Kalman Filter Estimate of the Uncertain Parameter in the System Dynamics Determined with Increased Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.001 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
True Value ——
Estimated Value

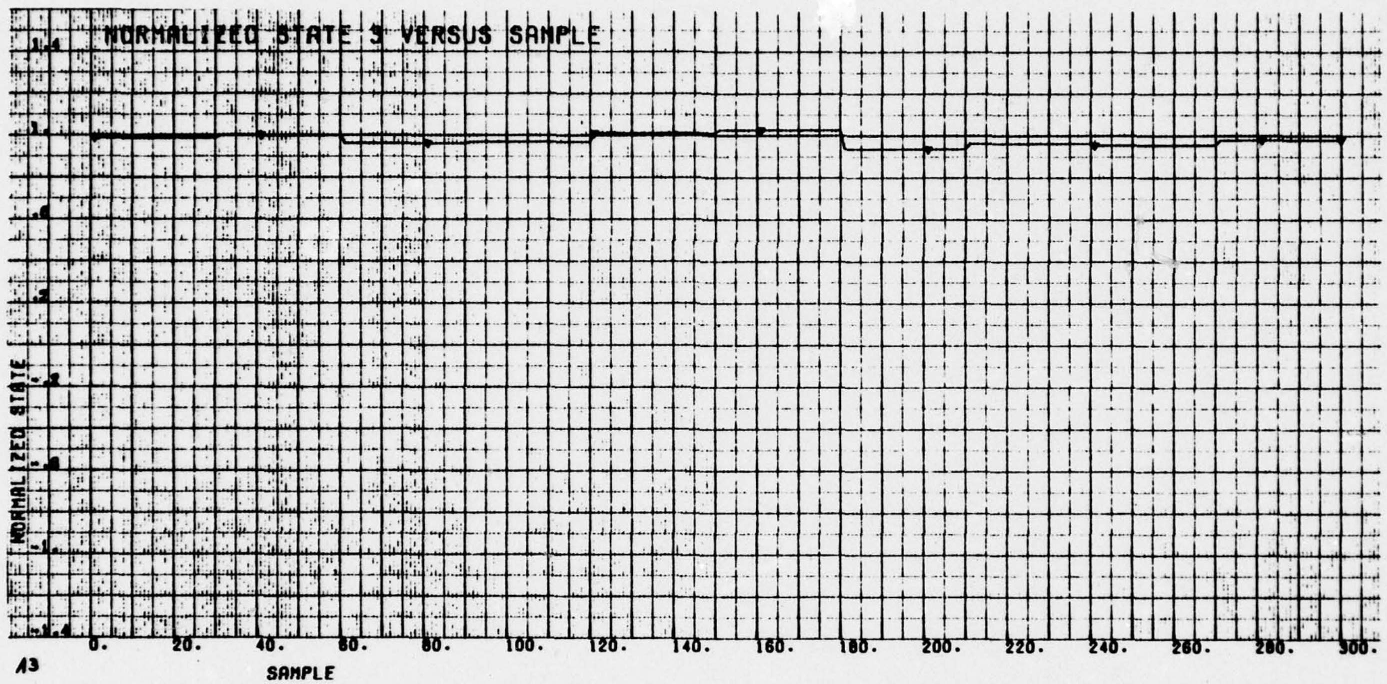

Figure 70. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics Determined with Increased Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Value - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Initial Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$

Sample Rate - 0.1 sec
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

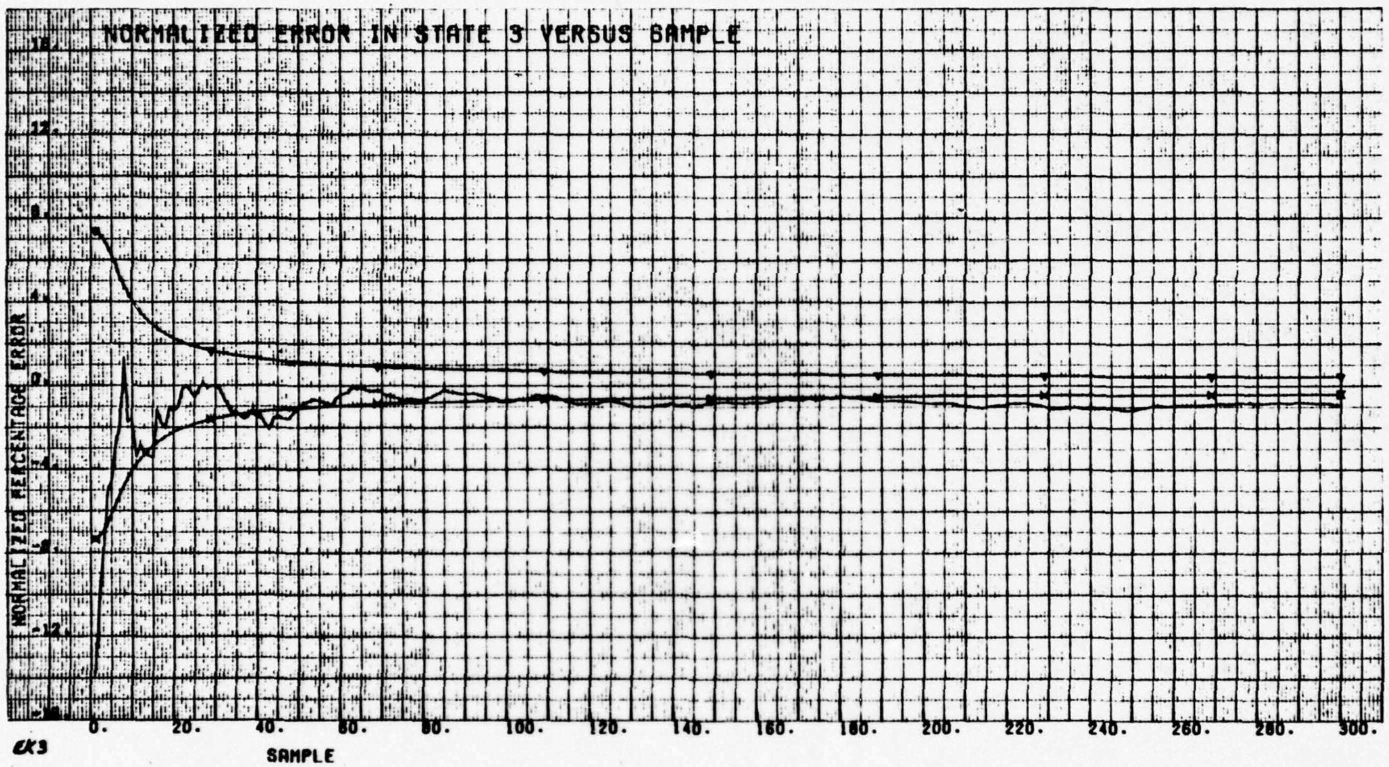

Figure 71. Extended Kalman Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with Increased Measurement Noise.

Parameter Value - 7.4 rad/sec
Initial Value - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.001 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

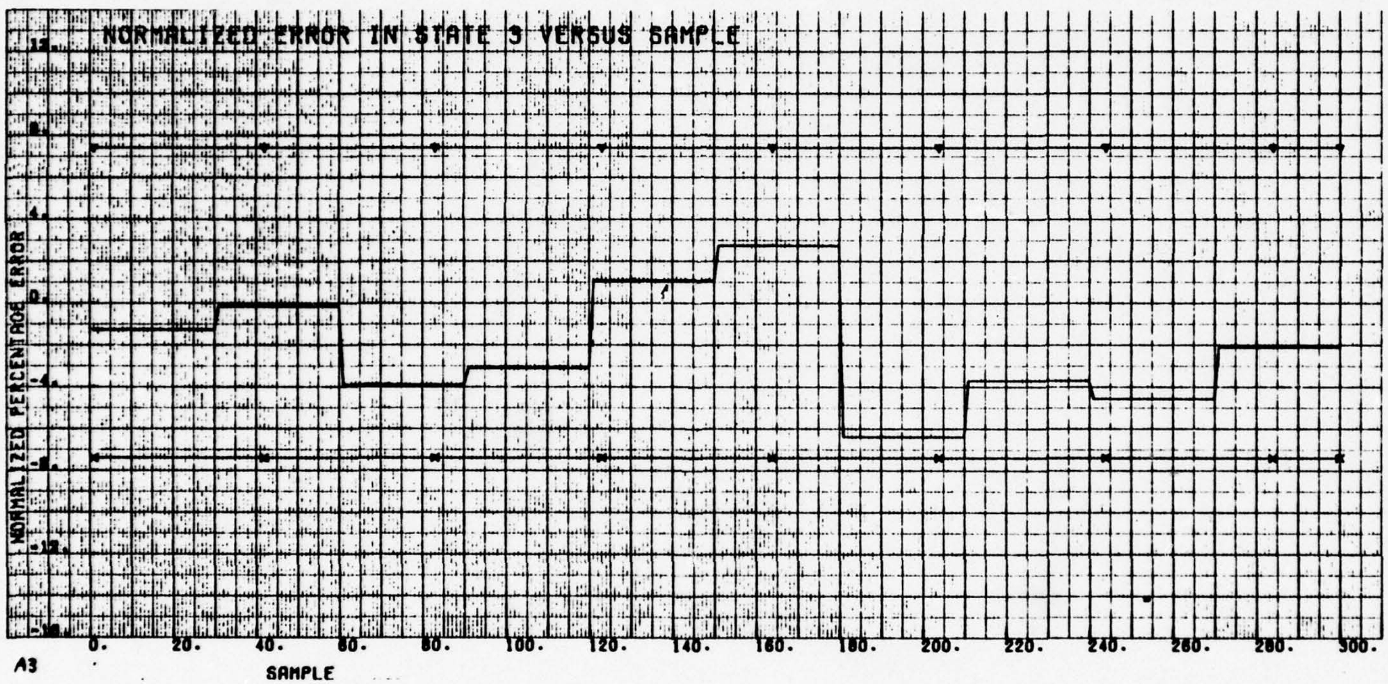

Figure 72. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with Increased Measurement Noise.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$

Sample Rate - 0.1 sec
Data Missing for Samples 100 thru 104
True State ——
Estimated State —▽—

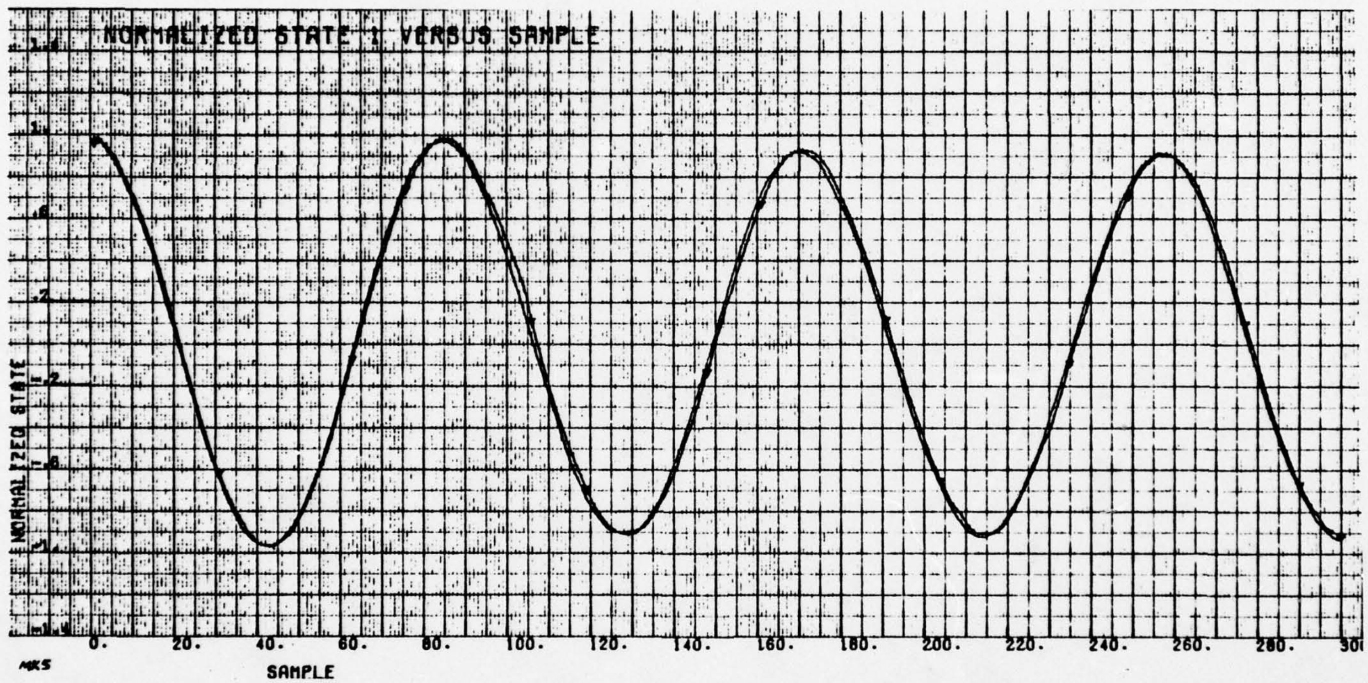

Figure 73. Modified Kalman Filter Estimate of State 1 Determined with Missing Measurement Data.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Data Missing for Samples 100 thru 104
True State ——
Estimated State —▽—

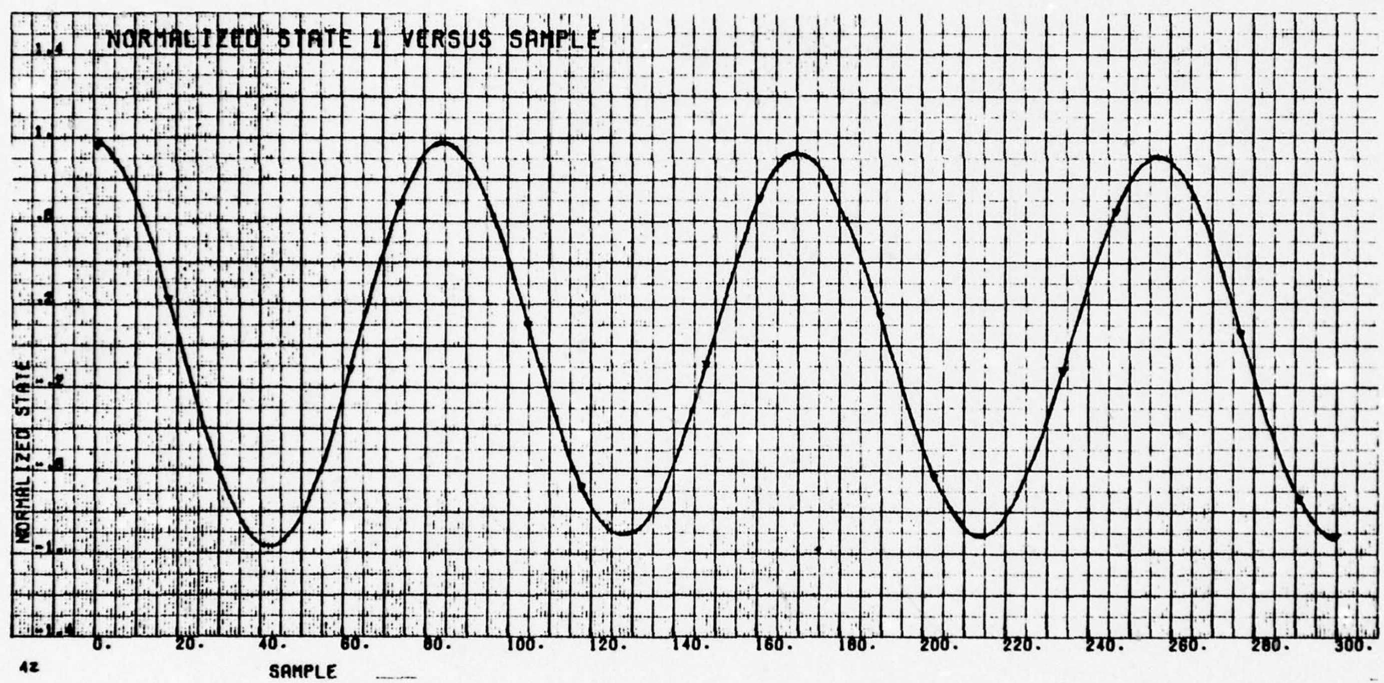

Figure 74. Adaptive Filter Estimate of State 1 Determined with Missing Measurement Data.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
Data Missing for Samples 100 thru 104
Estimate State ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

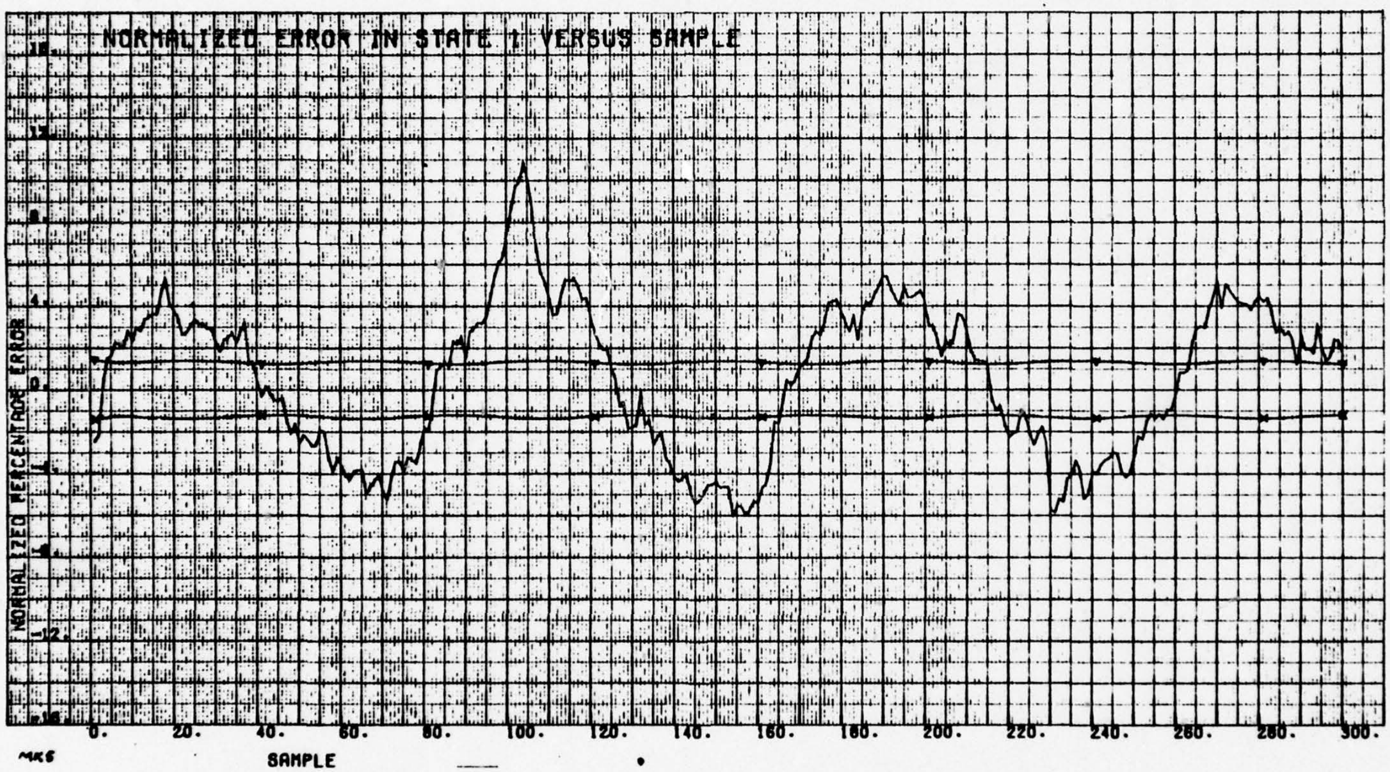

Figure 75. Modified Kalman Filter Estimate Error in State 1 Determined with Missing Measurement Data.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Data Missing for Samples 100 thru 104
Estimate State ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

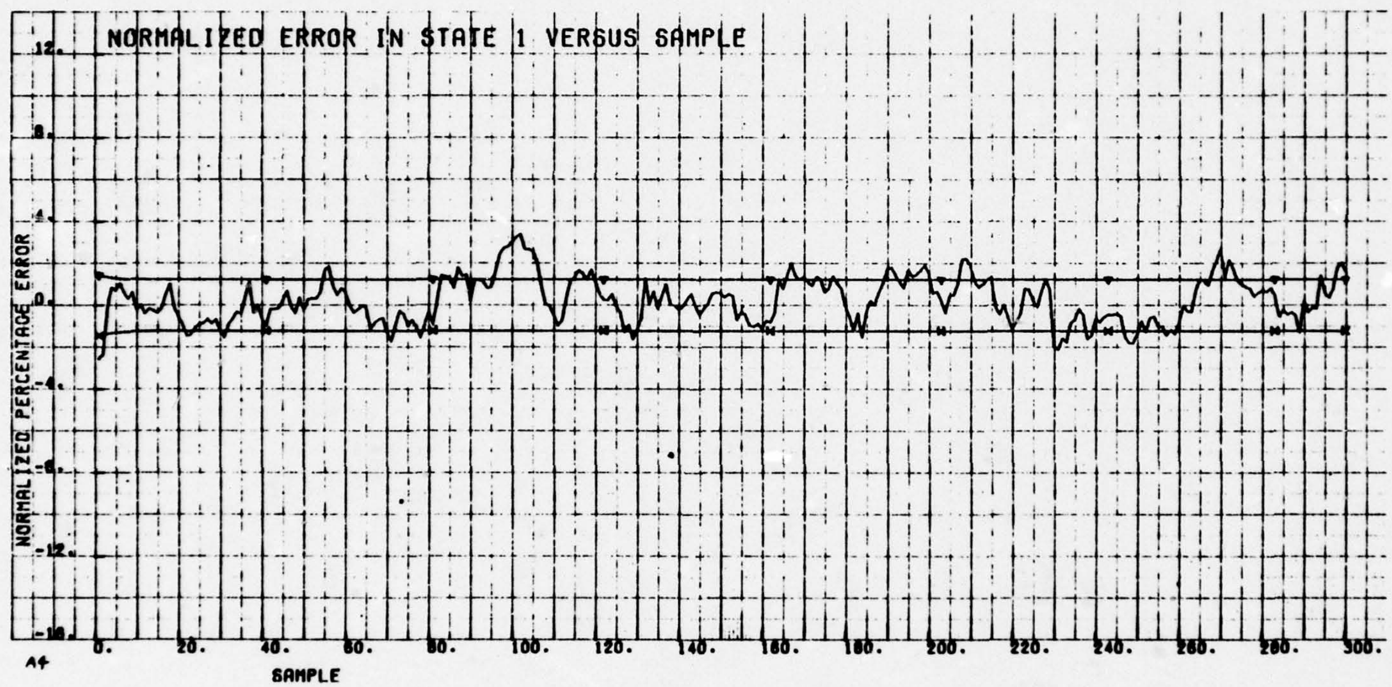

Figure 76. Adaptive Kalman Filter Estimate Error in State 1 Determined with Missing Measurement Data.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
Data Missing for Samples 100 thru 104
True State ——
Estimated State —∇—

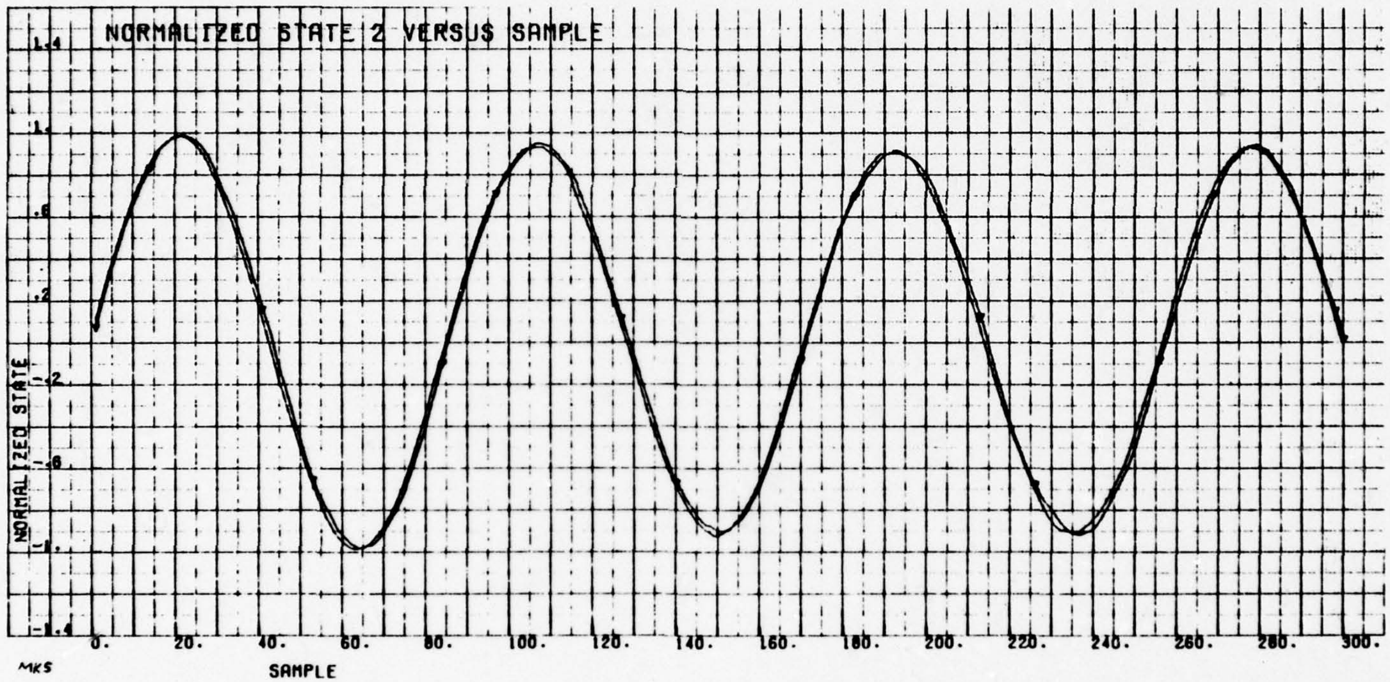

Figure 77. Modified Kalman Filter Estimate of State 2 Determined with Missing Measurement Data.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Data Missing for Samples 100 thru 104
True State ——
Estimated State —▽—

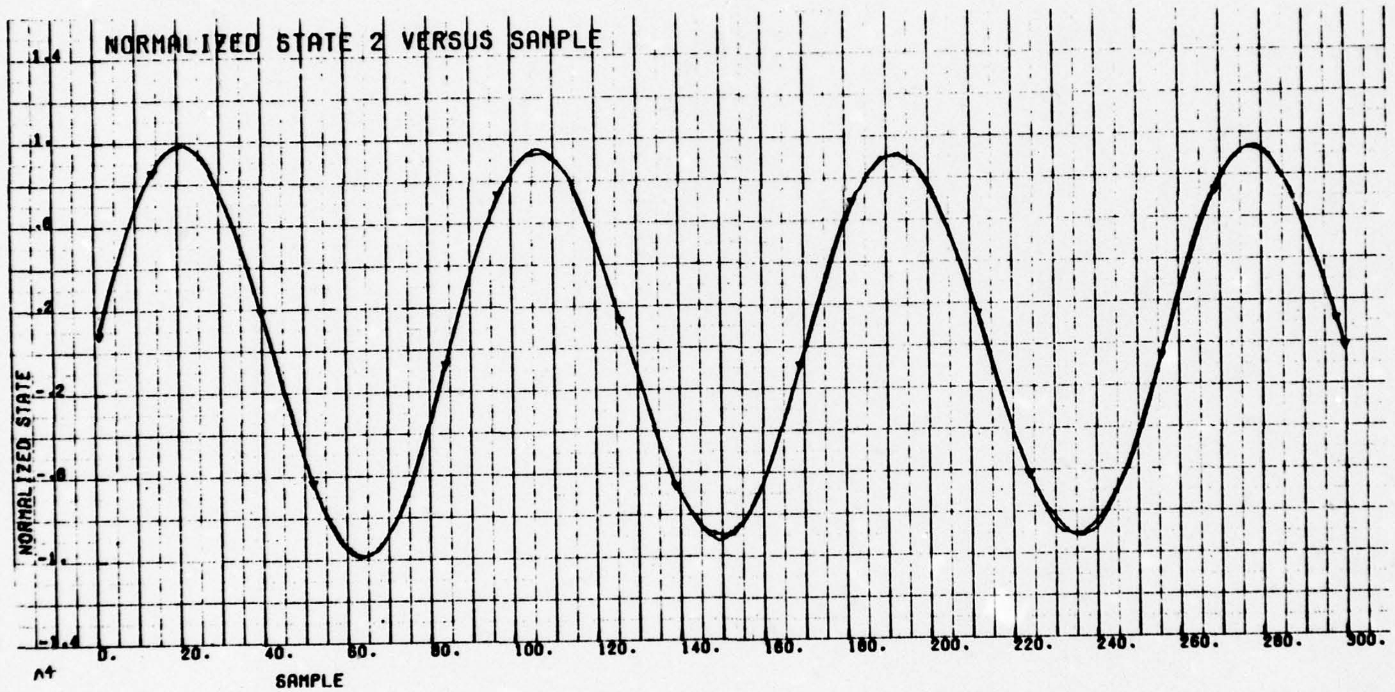

Figure 78. Adaptive Kalman Filter Estimate of State 2 Determined with Missing Measurement Data.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$

Sample Rate - 0.1 sec
Data Missing for Samples 100 thru 104
Estimate State ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

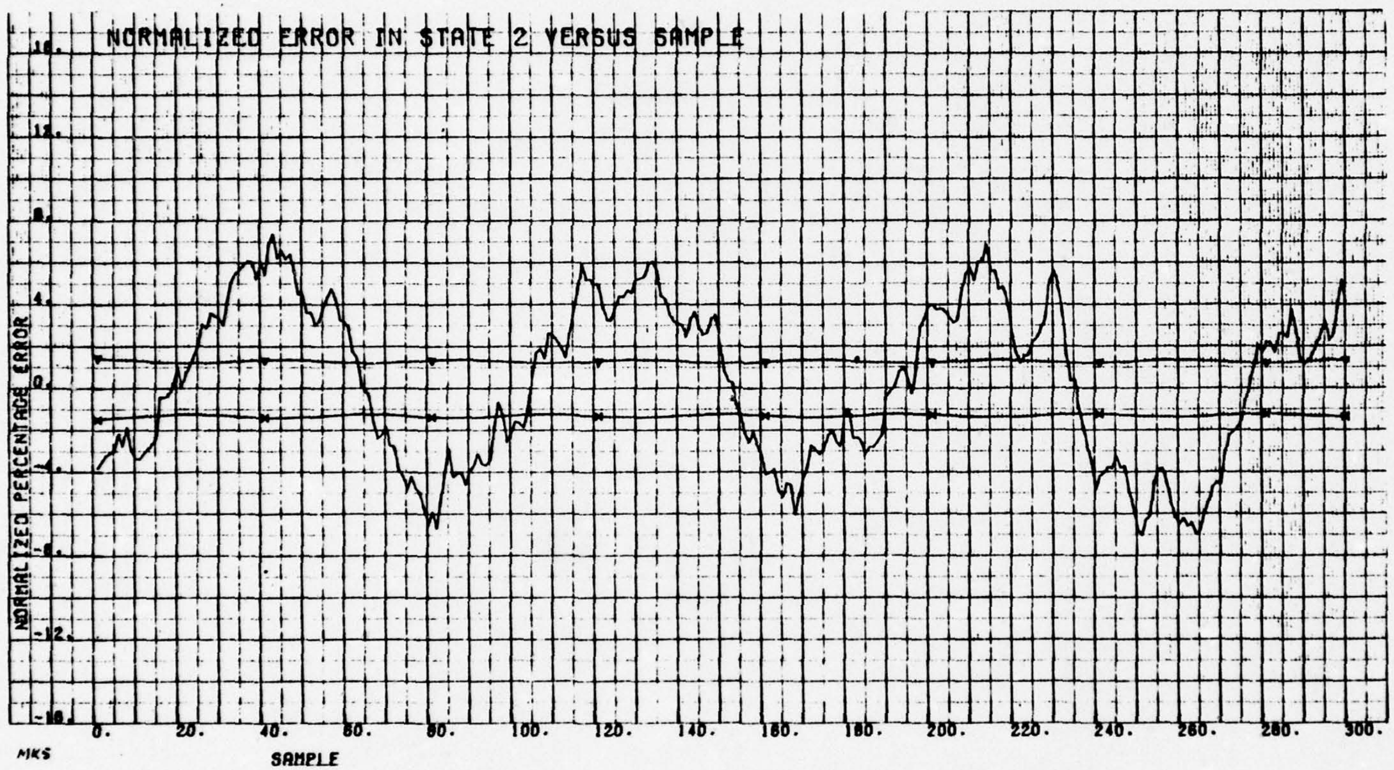

Figure 79. Modified Kalman Filter Estimate Error in State 2 Determined with Missing Measurement Data.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Data Missing for Samples 100 thru 104
Estimate State ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

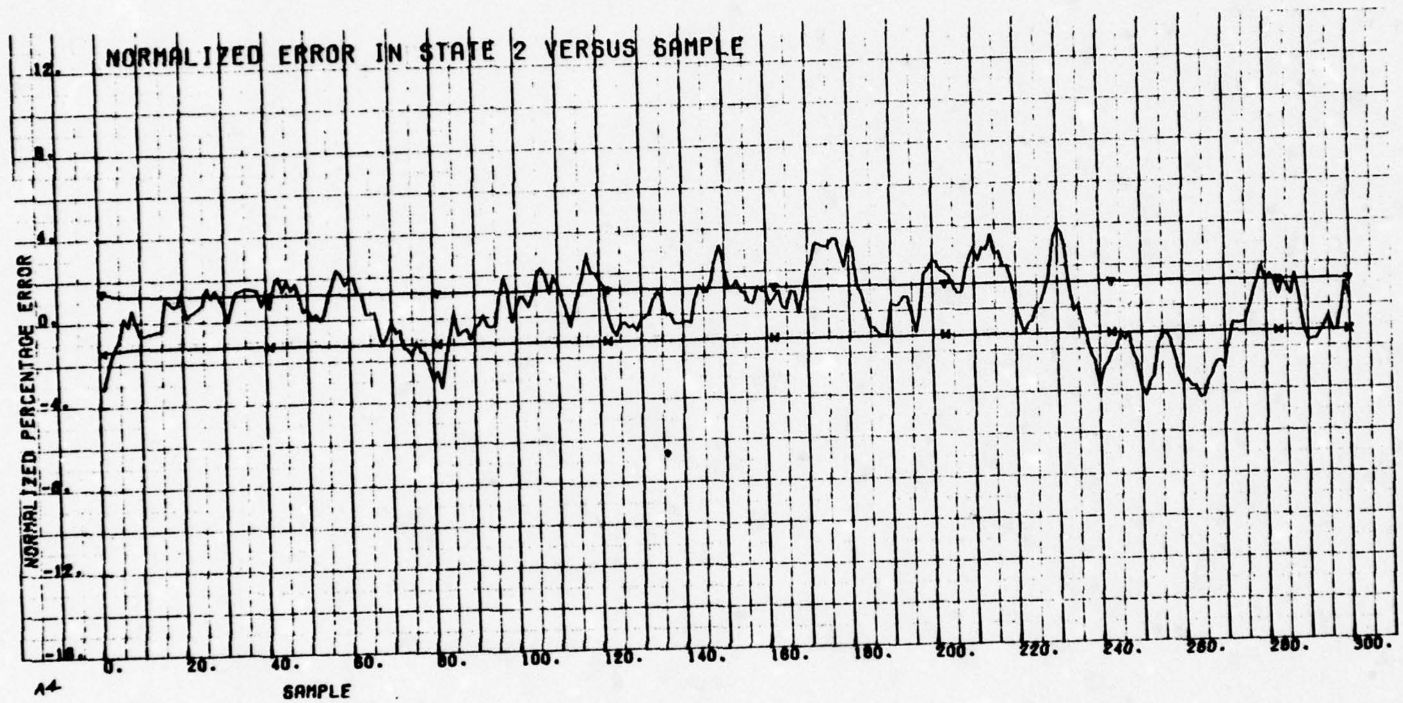

Figure 80. Adaptive Filter Estimate Error in State 2 Determined with Missing Measurement Data.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Data Missing for Samples 100 thru 104
True Value ——
Estimated Value

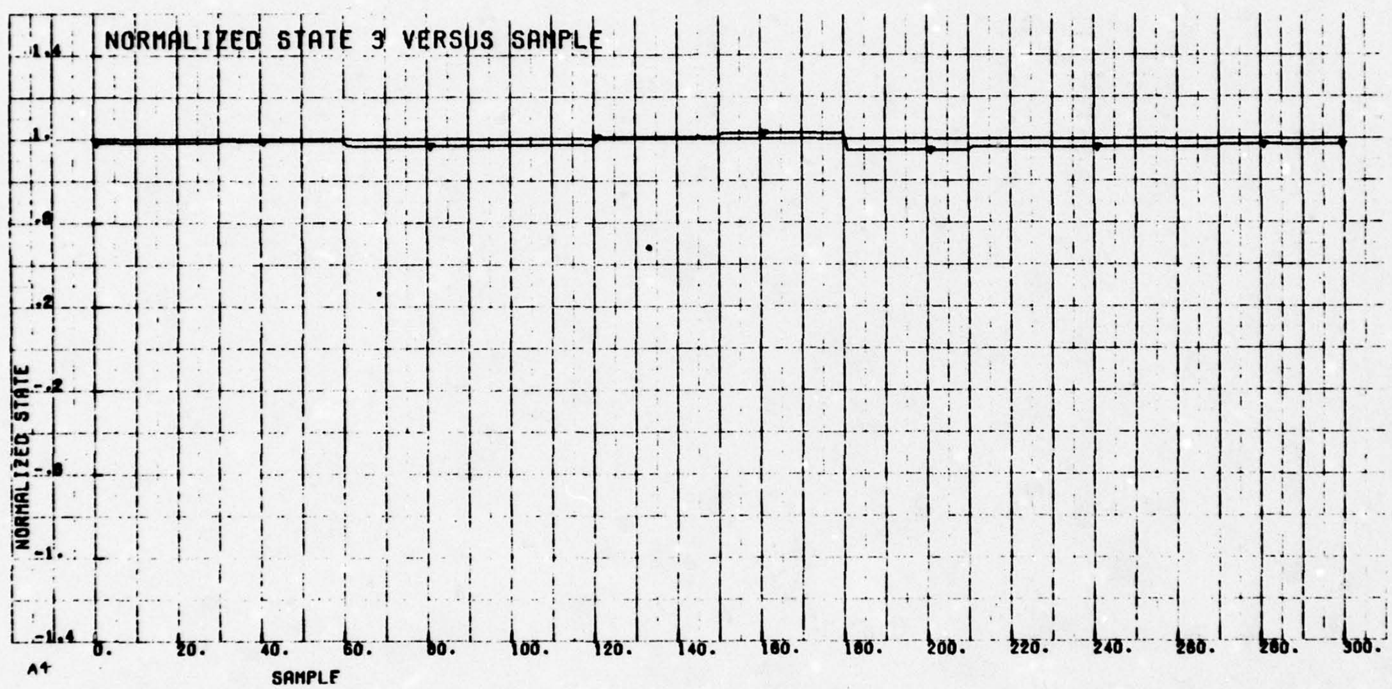

Figure 81. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics Determined with Missing Measurement Data.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Data Missing for Samples 100 thru 104
Estimate Error ——
Plus Estimate Standard Deviation -▽-
Minus Estimate Standard Deviation -X-

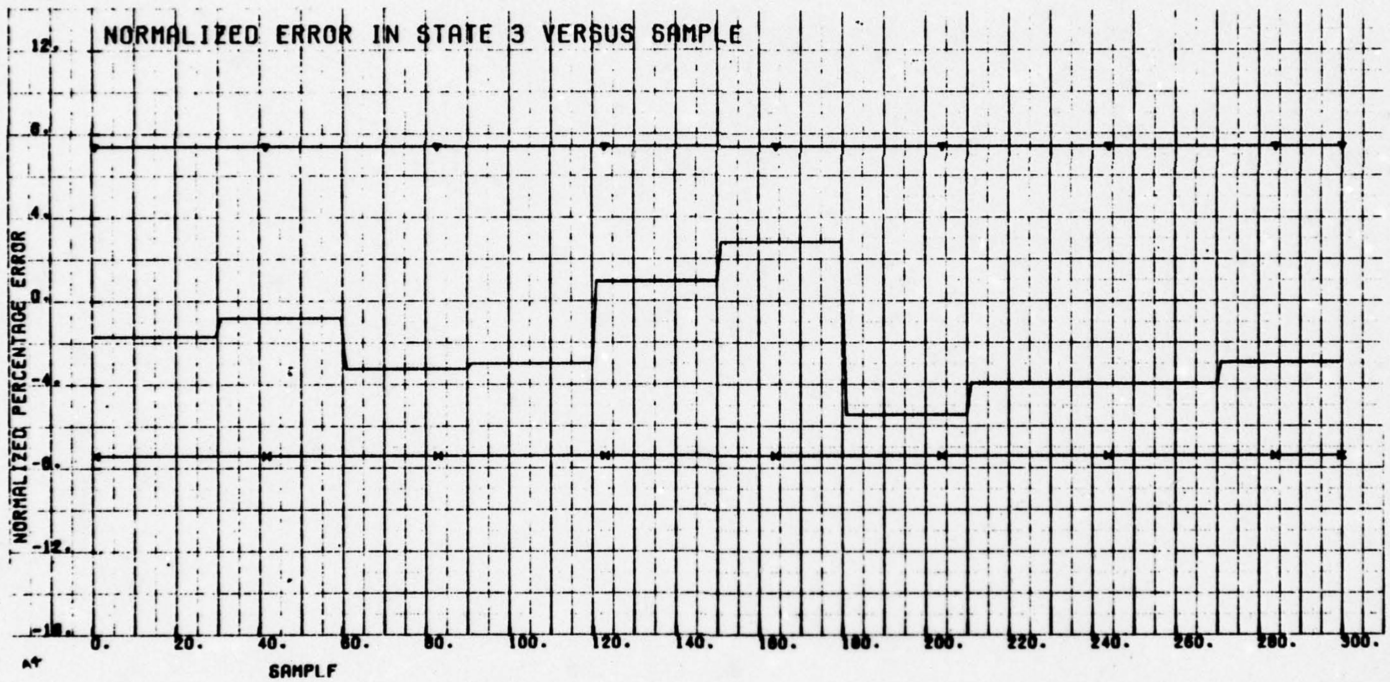

Figure 82. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with Missing Measurement Data.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Measurement Data at Sample 100 Multiplied by 10
True State ——
Estimated State —▽—

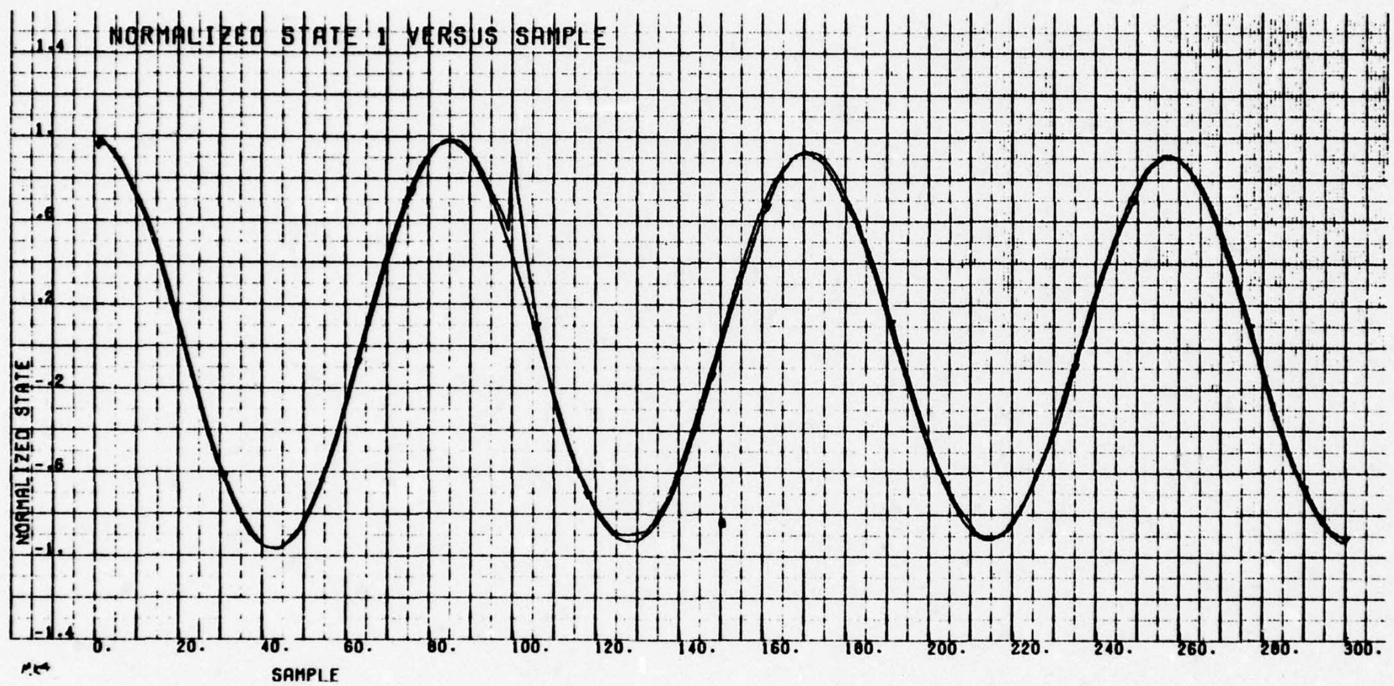

Figure 83. Modified Kalman Filter Estimate of State 1 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Measurement Data at Sample 100 Multiplied by 10
True State ——
Estimated State

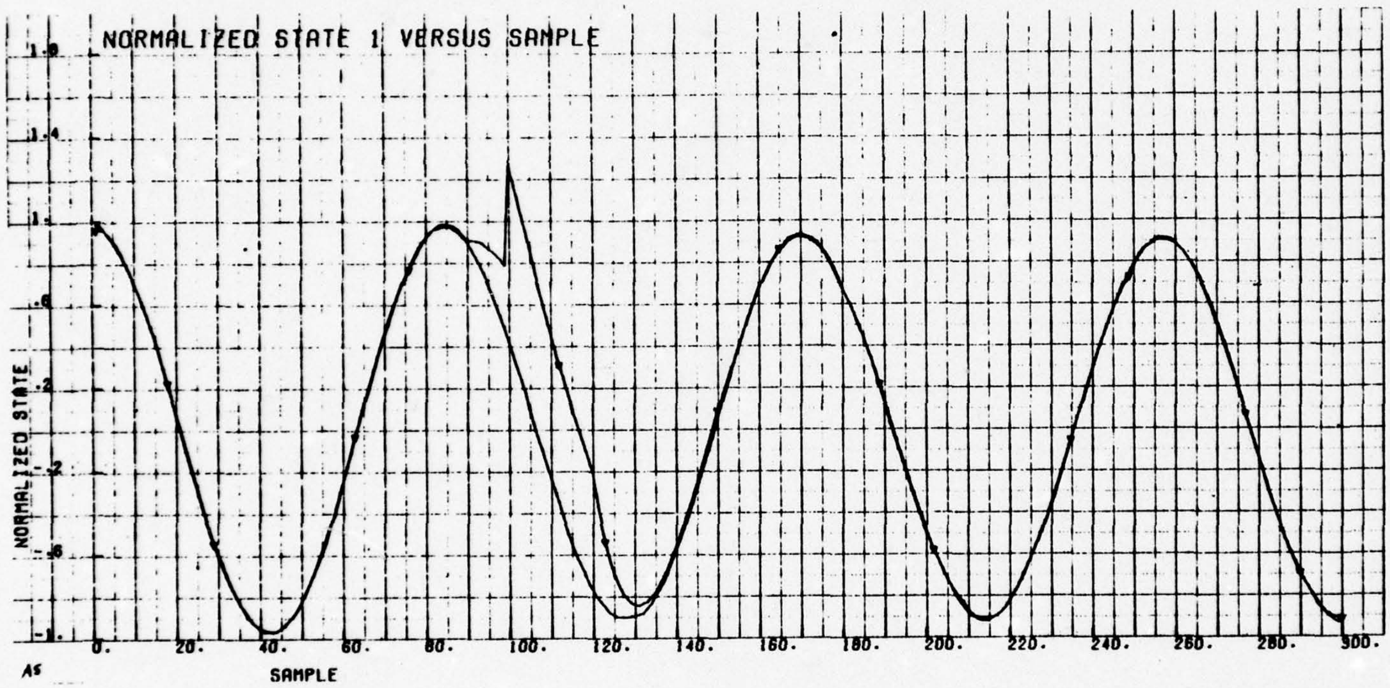

Figure 84. Adaptive Filter Estimate of State 1 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Measurement Data Sample 100 Multiplied by 10

Estimate State ——
Plus Estimate Standard Deviation —∇—
Minus Estimate Standard Deviation —X—

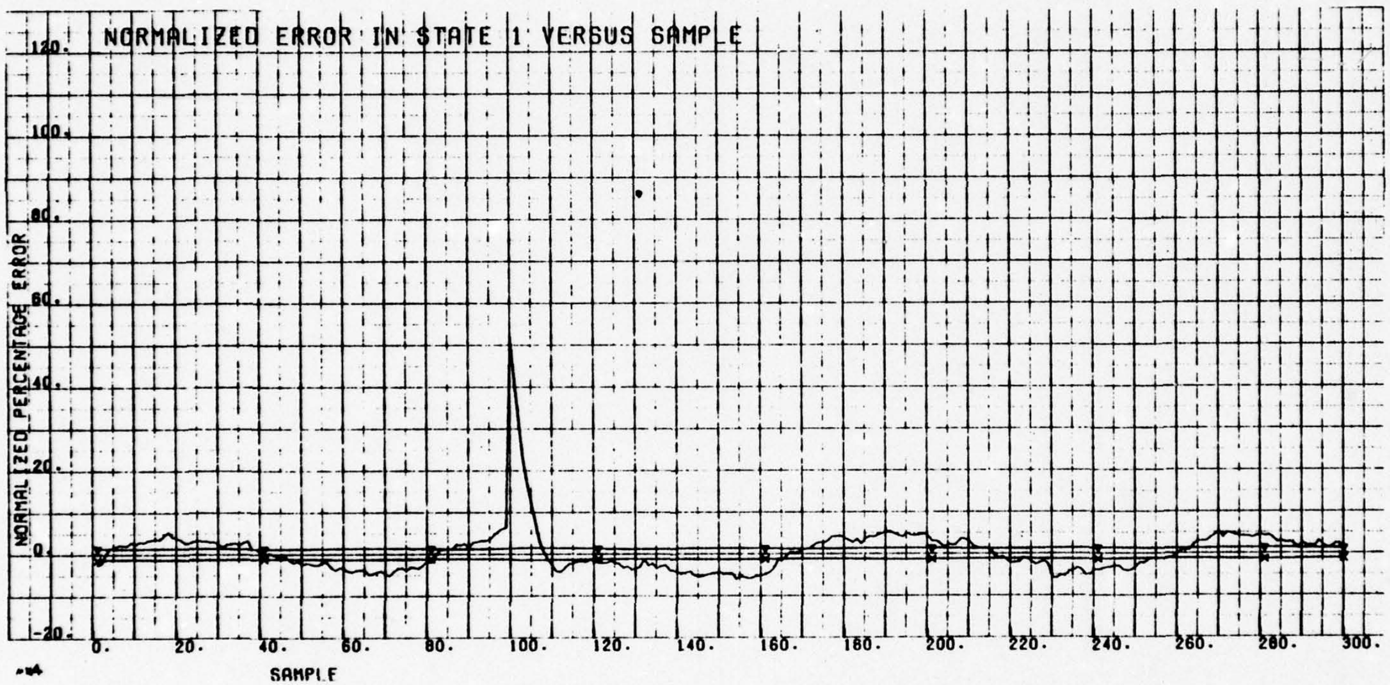

Figure 85. Modified Kalman Filter Estimate Error in State 1 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.66 rad/sec
Initial Estimate - 0.63 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Measurement Data Sample 100 Multiplied by 10

Estimate State ——
Plus Estimate Standard Deviation —⊽—
Minus Estimate Standard Deviation —X—

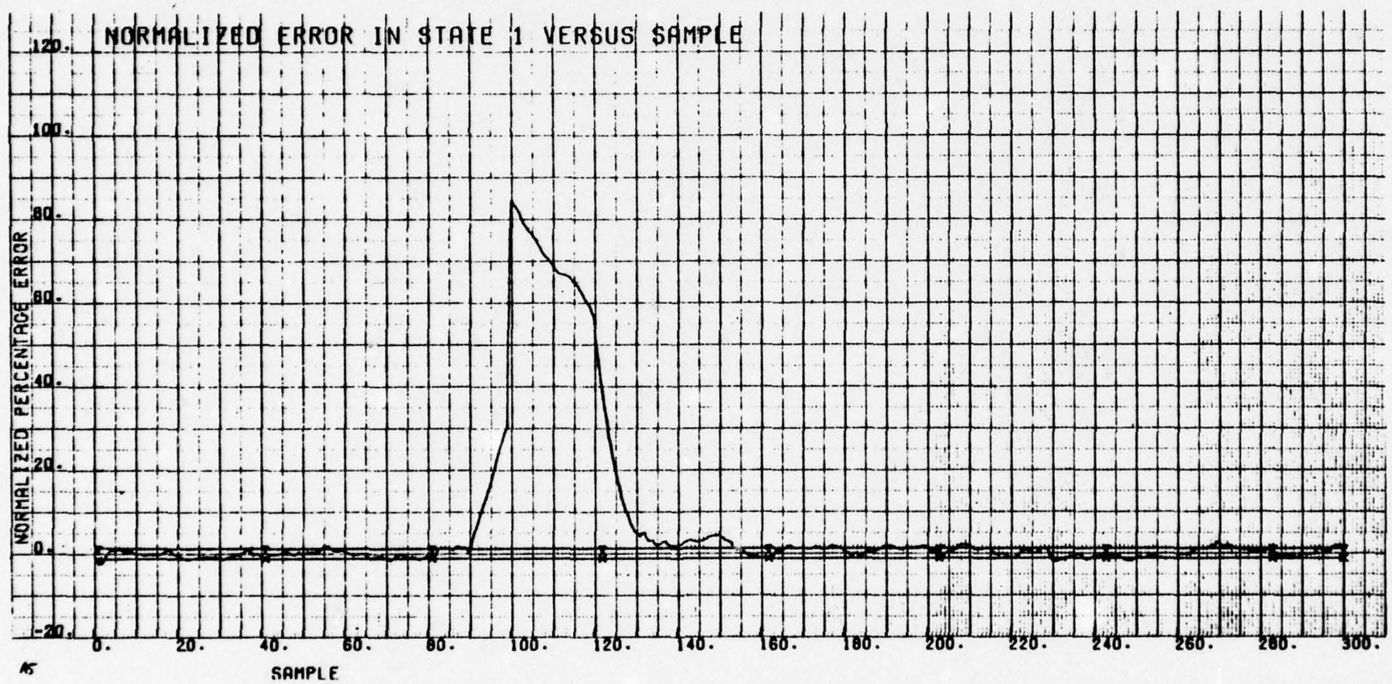

Figure 86. Adaptive Filter Estimate Error in State 1 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec

Measurement Data at Sample 100 Multiplied by 10
True State ——
Estimated State

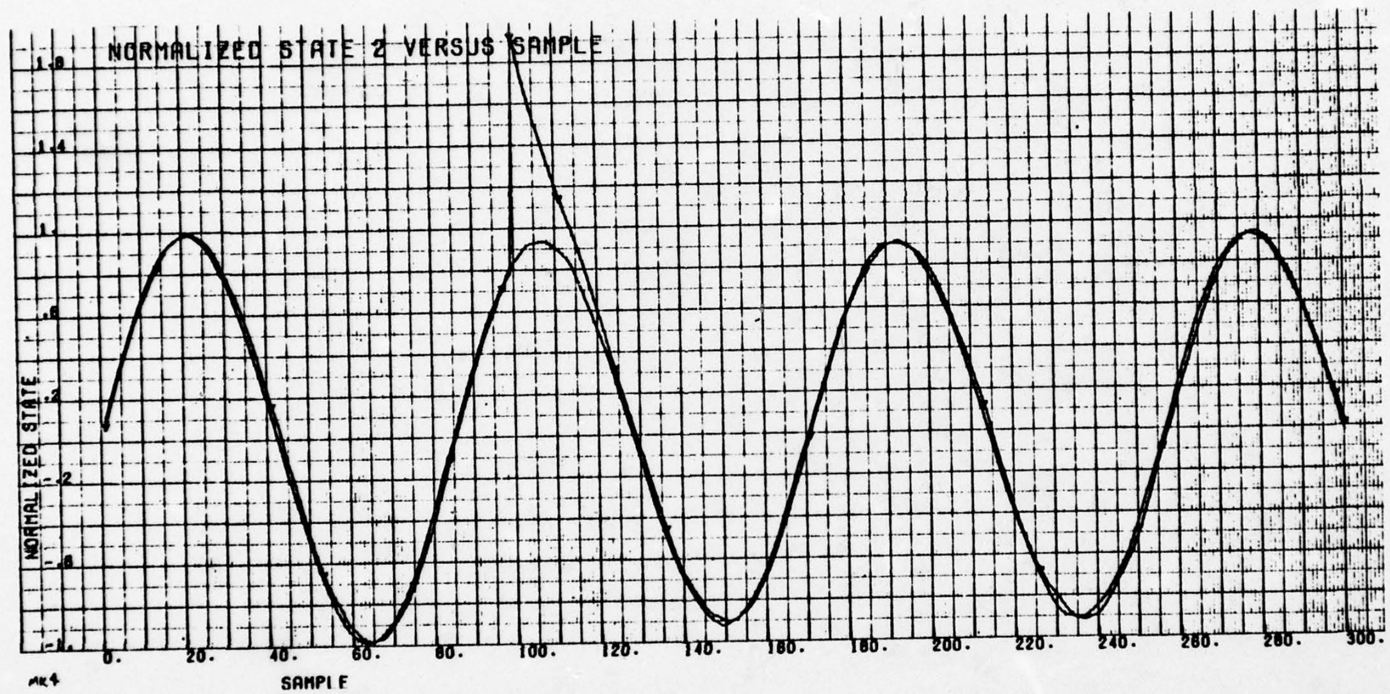

Figure 87. Modified Kalman Filter Estimate of State 2 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Measurement Data at Sample 100 Multiplied by 10
True State ——
Estimated State —∇—

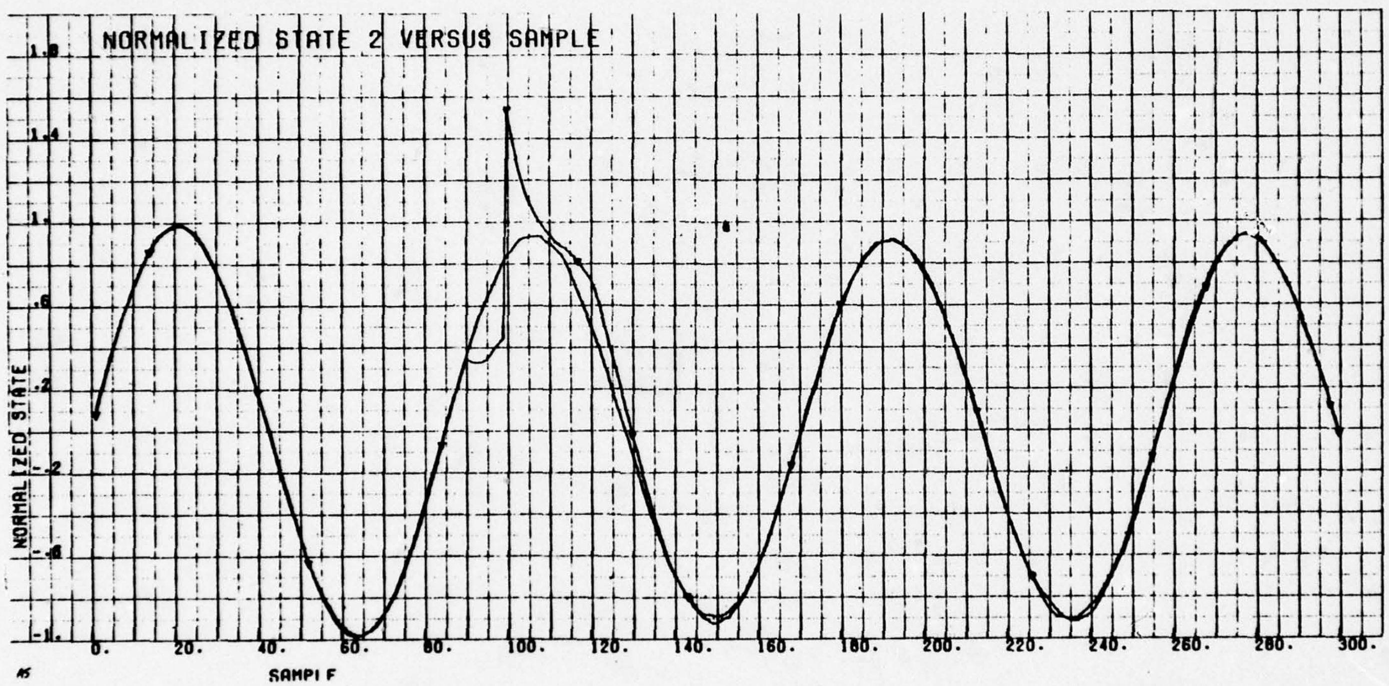

Figure 88. Adaptive Filter Estimate of State 2 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(\text{rad/sec})^2$
System Noise Covariance - 0.00001 $(\text{rad/sec})^2$
Measurement Noise Covariance - 0.0005 $(\text{rad/sec})^2$
Sample Rate - 0.1 sec
Measurement Data Sample 100 Multiplied by 10

Estimate State ——
Plus Estimate Standard Deviation —∇—
Minus Estimate Standard Deviation —X—

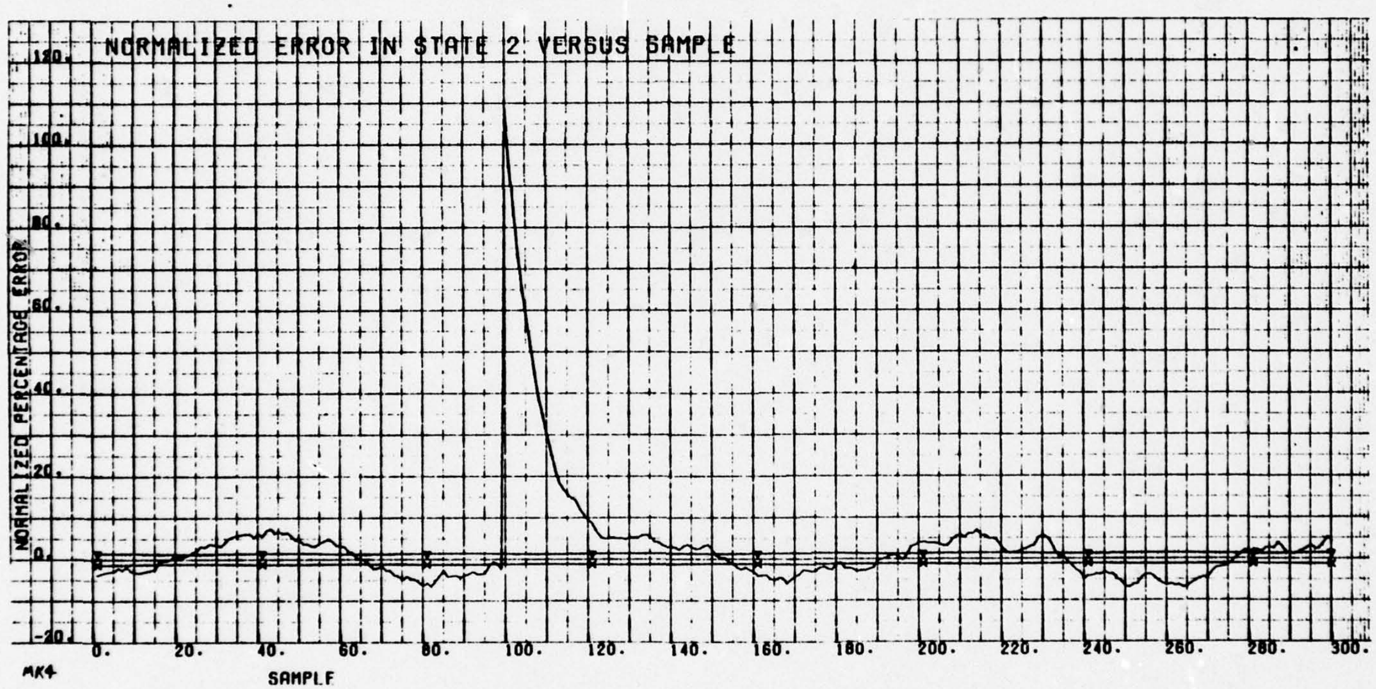

Figure 89. Modified Kalman Filter Estimate Error in State 2 Determined with an Erroneous Measurement Data Spike.

Initial State - 0.03 rad/sec
Initial Estimate - 0.0 rad/sec
Normalizing Constant - 0.66 rad/sec
Initial State Covariance - 0.0001 $(rad/sec)^2$
System Noise Covariance - 0.00001 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30
Measurement Data Sample 100 Multiplied by 10

Estimate State ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

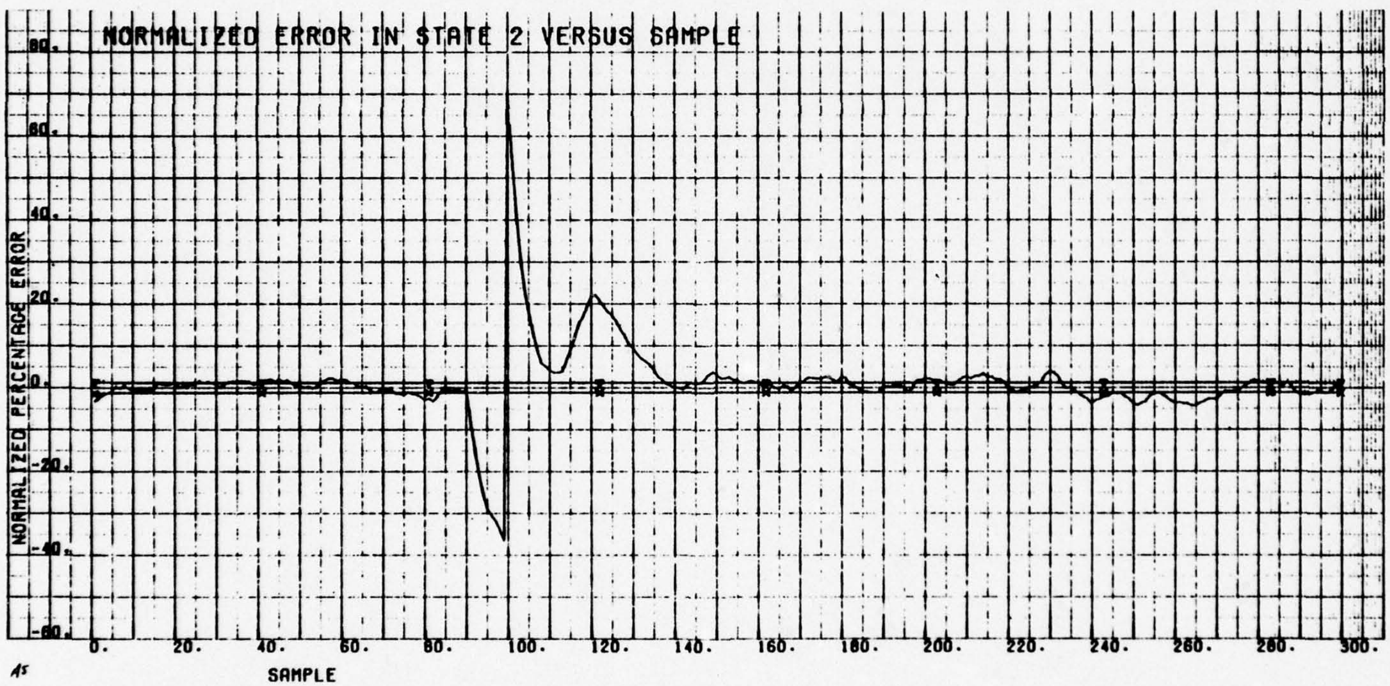

Figure 90. Adaptive Filter Estimate Error in State 2 Determined with an Erroneous Measurement Data Spike.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec

Samples Used for Parameter Estimate - 30
Measurement Data at Sample 100 Multiplied by 10
True Value ——
Estimated Value —▽—

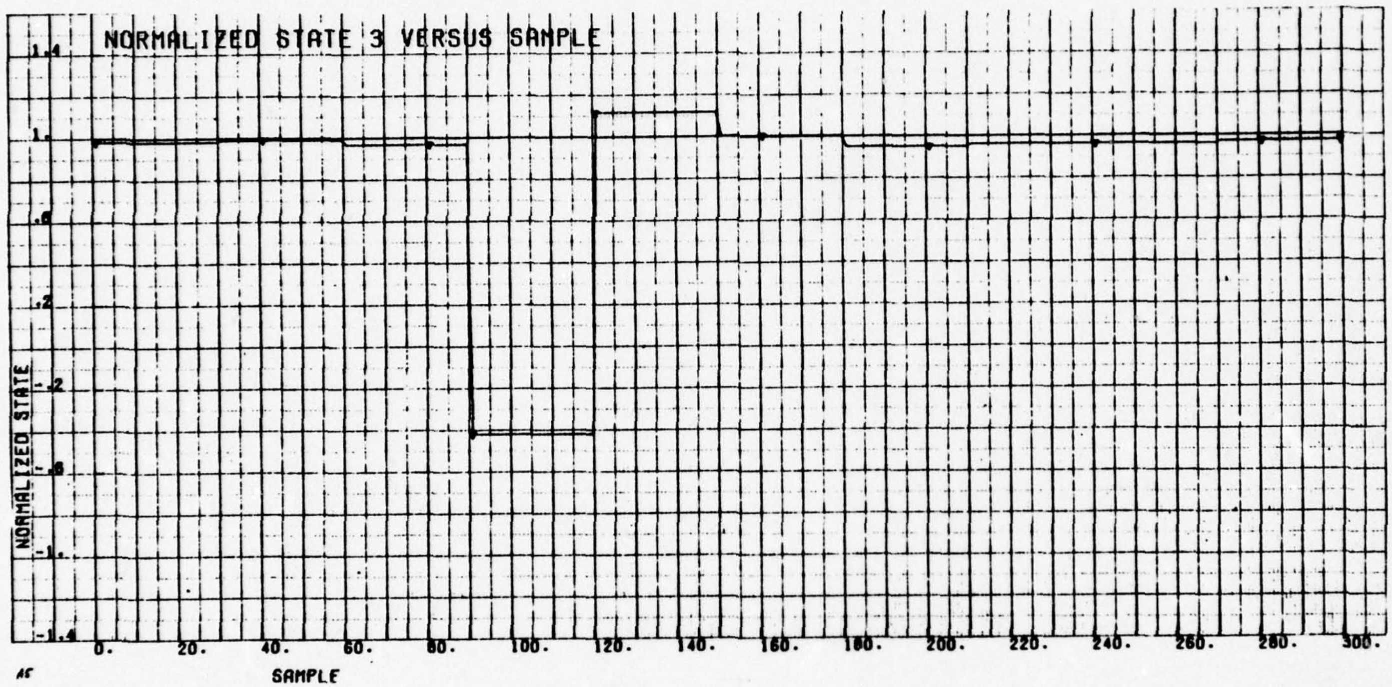

Figure 91. Adaptive Filter Estimate of the Uncertain Parameter in the System Dynamics Determined with an Erroneous Measurement Data Spike.

Parameter Value - 7.4 rad/sec
Initial Estimate - 6.3 rad/sec
Normalizing Constant - 7.4 rad/sec
Parameter Covariance - 0.3 $(rad/sec)^2$
Measurement Noise Covariance - 0.0005 $(rad/sec)^2$
Sample Rate - 0.1 sec
Samples Used for Parameter Estimate - 30

Measurement Data at Sample 100 Multiplied by 10
Estimate Error ——
Plus Estimate Standard Deviation
Minus Estimate Standard Deviation

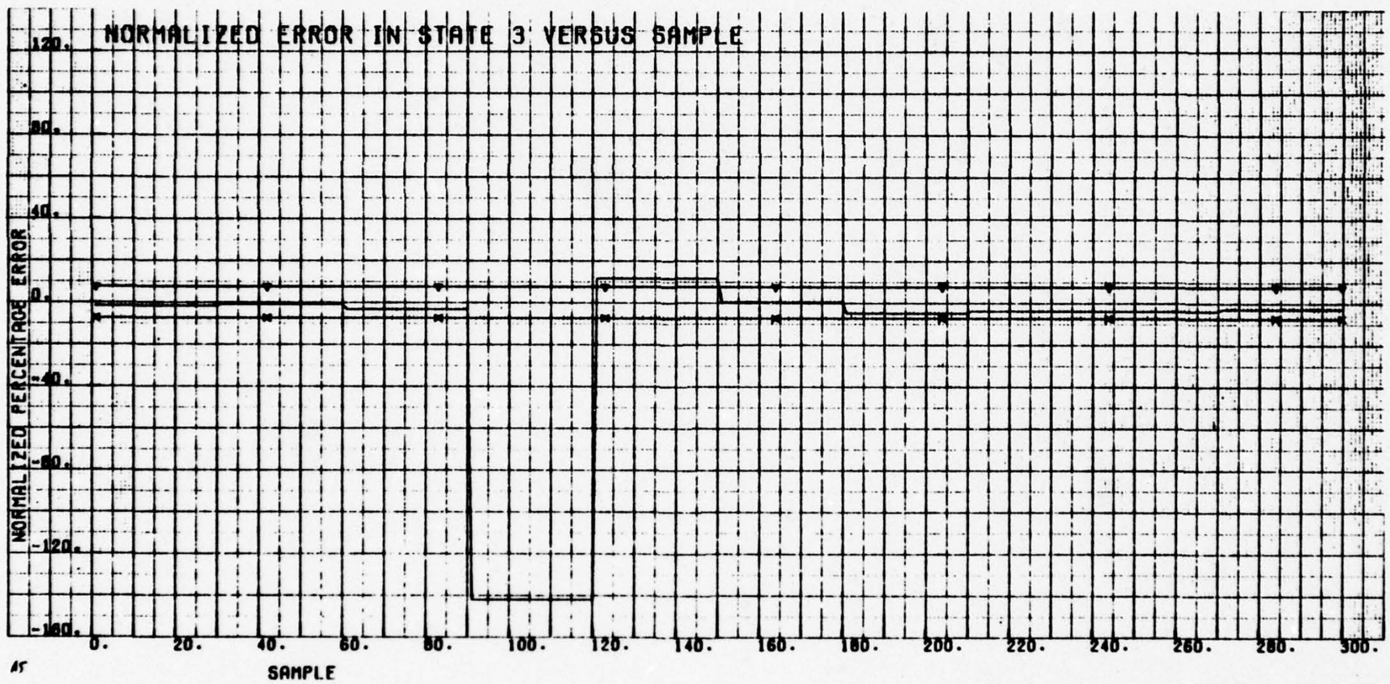

Figure 92. Adaptive Filter Estimate Error in the Uncertain Parameter in the System Dynamics Determined with an Erroneous Measurement Data Spike.

CHAPTER 6

CONCLUSION

This chapter provides a summary of the research results and recommendations for future investigations.

6.1 SUMMARY OF RESULTS

In the dissertation, two filter techniques are proposed for estimation of the states of a linear dynamic system using noise-corrupted measurements when parameters in the system model are not exactly known. The first technique is a modification of the Kalman filter. The method incorporates the parameter uncertainties in the filter by adding additional terms to the Kalman gain equation. In general, when the parameter uncertainties exist in the state equation, the effect of these terms is to increase the filter gain since the measurement is a more accurate assessment of the state than the predicted state. Conversely, when the parameter uncertainties exist in the measurement equation, the gain is decreased since the predicted state is a more accurate appraisal relative to the measurement.

The second technique is an adaptive method based on the properties of the innovation sequence of the optimum

Kalman filter. This method makes a maximum likelihood Bayesian estimate of the uncertain parameters operating iteratively in conjunction with a Kalman filter. The Kalman filter uses the current estimate of the parameters to generate the innovations and other information utilized by the Bayesian estimator of the parameters. With the improved values for the parameters, the Kalman filter is again operated to regenerate the state estimates.

Numerical results for an example problem were generated for the modified Kalman filter and the adaptive filter. For comparison, data was also generated for the Kalman and extended Kalman filters. In general, the modified Kalman filter produced a more accurate estimate than the Kalman filter in about the same computational time. The extended Kalman, however, provided a significantly better estimate than either the Kalman or modified Kalman filter by estimating the uncertain parameter but did consume slightly more computational time. The accuracy of the adaptive filter was slightly less than that of the extended Kalman filter except during the initial sample period and the computational time was significantly greater. Since the computational time is only a few seconds, the larger computational time may not be a disadvantage for

the adaptive filter in some problems. In addition, the adaptive filter may be more useful since the extended Kalman filter is sometimes plagued by divergence. The basic reason that the estimate diverges from the actual state is that the gain in the filter algorithm approaches zero too rapidly. The multiplication of the gain and the innovation (i.e., the measurement residual) in the filter algorithm thus has a negligible effect on the estimate even though the innovation may be quite large. Hence, the estimate becomes decoupled from the observation sequence and is not affected by the increasing measurement residuals. On the other hand, by designing the adaptive filter to provide a maximum likelihood estimate, the measurement residuals form the basis for the estimate. Operation of the filter attempts to minimize the measurement residuals. Consequently, the filter attempts to adjust and correct the estimates with increasing measurement residuals and thus ensure filter stability. The proof of this behavior is not considered here, but could be addressed in future studies.

When the level of measurement noise was increased or decreased, the errors in the estimates generated by the four filters tended to increase or decrease correspondingly. However, the effect was more significant on the results of

the Kalman and modified Kalman filters. The extended Kalman filter and adaptive filter showed relatively little difference with the variation in measurement noise level.

The effect of missing data on the modified Kalman filter was analyzed and found to increase the estimation error. The increase, however, was only temporary in the example and became equivalent to results generated with the complete set of data after approximately 20 samples. The effect of the data gap on the adaptive filter was relatively small and differed little from results generated with the full complement of data.

An erroneous data spike was also analyzed and produced a significant increase in the estimation error of the modified Kalman filter, which subsequently affected approximately 25 state estimates. After this period, the estimates returned to the values generated without the data spike. Because of the iterative nature of the adaptive filter, the error in the adaptive filter estimates prior to the measurement data spike was increased and the estimation error was abnormally higher for 50 to 60 estimates after the data spike. The estimates did, however, return to the values generated without the data spike. Due to the dramatic effect of the data spike on the

estimation error, practical use of the adaptive filter may require a pre-filter to limit the effects of data spikes. Additional experimental or data analyses, however, would be required to develop and establish the design of the pre-filter or techniques which treat effectively data spikes.

Tables 1 through 4 summarize the numerical results. In general, the Kalman filter can be applied to the problem of state estimation for systems with uncertain parameters provided that the estimates meet the desired accuracy requirements. If greater accuracy is required, then the modified Kalman, the extended Kalman or the adaptive filter might be considered. The desirability of each filter, however, must be addressed on a case by case basis. The extended Kalman filter can produce greater accuracy than the modified Kalman filter provided that divergence of the extended Kalman filter estimates is avoided. The adaptive filter can produce about the same accuracy; however, the computational time of the filter may be greater.

6.2 RECOMMENDATIONS FOR FUTURE INVESTIGATION

There are several areas of investigation that could be considered for future effort. These topics are described in the following paragraphs.

First, the innovation sequence of the Kalman filter was used in the adaptive filter to generate state estimations based on a predefined schedule for the measurements. If the frequency of measurements is increased or decreased, the magnitude of values of the innovation sequence and its accompanying covariance would be expected to increase or decrease correspondingly. This leads to a change in the accuracy of the state estimates. Since the taking and processing of measurements represent a certain effort or cost, a tradeoff exists between the estimate accuracy and the cost of taking the measurements. Thus, by utilizing the innovation sequence, an adaptive method might be developed that determines the measurement schedule while maintaining a certain degree of estimation accuracy.

Secondly, the adaptive filter consumed a relatively larger amount of computational time than other filters analyzed. Therefore, methods for reducing the computational time could be investigated. For example, certain quantities might be precomputed, or summetry in matrix relationships might be exploited.

Third, the divergence problem, or instability, of the extended Kalman filter was noted in Chapter 5. Although some studies of stability have been performed

for the Kalman filter, studies of the stability of the extended Kalman filter are almost nonexistent. Therefore, study efforts on the extended Kalman filter stability should be performed to guide the determination of appropriate applications for the extended Kalman filter. In addition, stability of the adaptive filter utilizing the maximum likelihood estimator could also be addressed in such a study.

Finally, the simultaneous problem of optimum control and state estimation for systems with certain parameters might be considered. In this study, only the problem of estimation was addressed. Probably the easiest as well as an important case of optimal control to be examined would be linear feedback control. Another area of potential study is the development of a separation theorem between the optimal control and the estimate for systems with uncertain parameters. This study might initially consider the report by Snyder (1977).

TABLE 1

Filter Comparison Summary

Filter	Kalman	Modified Kalman	Extended Kalman	Adaptive
Maximum State Estimate Error	~8%	~8%	~2%	~3%
Final Parameter Estimate (True Value = 7.4)	NA	NA	7.33	7.18
Computational Time (300 Data Samples)	1.86 sec	1.90 sec	2.34 sec	9.56 sec

NA - Not Applicable

TABLE 2

Effect of an Initial Zero Parameter Estimate on Extended Kalman and Adaptive Filter Estimates

Filter	Extended Kalman	Adaptive
State Estimates Affected	~50	0
Maximum Increased In State Estimate Error	~10%	0%
Parameter Estimates Affected	~70	0
Final Parameter Estimate (True Value = 7.4)	7.31	7.18

TABLE 3

Effect of Measurement Noise

Filter	Maximum State Estimate Error			
	Kalman	Modified Kalman	Extended Kalman	Adaptive
Nominal Noise (Covariance = 0.0005)	~8%	~6%	~2%	~3%
Reduced Noise (Covariance = 0.00025)	~6%	~5%	~2%	~3%
Increased Noise (Covariance = 0.001)	~11%	~9%	~3%	~4%

TABLE 4

Effect of Data Aberrations

Filter	Missing Data (5 Data Samples)		Data Spike	
	Modified Kalman	Adaptive	Modified Kalman	Adaptive
Estimates Affected	~10	~6	~25	~55
Maximum Increased In State Estimate Error	~3%	< 1%	~100%	~80%

APPENDIX A

DISCRETE REPRESENTATION OF CONTINUOUS DYNAMIC SYSTEMS

This appendix will develop the discrete system model that is equivalent to the continuous-time description. Assume that the continuous-time model of the system is given by the stochastic differential equation

$$dx = Fx\,dt + Bu\,dt + G\,d\beta \tag{A-1}$$

or the equivalent, through mathematically less precise, *linear differential equation*

$$\dot{x} = Fx + Bu + Gw \tag{A-2}$$

where w is white Gaussian noise and formally the derivative of Brownian motion β. The white noise is a zero-mean process with covariance $\underline{Q}(t)\ \delta(t-\tau)$ where $\underline{Q}(t)$ is chosen to duplicate the low frequency power spectral density of the actual noise entering the system. The statistics of the noise are expressed by the equations

$$E[w] = 0 \tag{A-3}$$

$$E\left[w(t_1)w^T(t_2)\right] = Q(t_1)\,\delta(t_1-t_2) \tag{A-4}$$

It is assumed that the measurements are taken at discrete instants and are of the form

$$z_i = H_i x_i + v_i \tag{A-5}$$

Furthermore, it is assumed that a digital computer will provide the control input, so that u(t) will be piecewise constant. That is, a measurement would be taken, the information processed, and a control input created and held constant until the following sample time.

The solution to Equation (A-1) or (A-2) is

$$x(t) = \Phi(t,t_i)x(t_i) + \int_{t_i}^{t} \Phi(t,\tau)B(\tau)u(\tau)dt$$

$$+ \int_{t_i}^{t} \Phi(t,\tau)G(\tau)dw(\tau) \tag{A-6}$$

where the state transition matrix $\Phi(t,t_i)$ satisfies the differential equation and initial condition

$$\frac{d}{dt}\Phi(t,t_i) = F(t)\Phi(t,t_i) \tag{A-7}$$

$$\Phi(t_i, t_i) = I \tag{A-8}$$

Since the control is held constant over a given sample period, the solution within a single sample period is

$$x(t) = \Phi(t, t_i)x(t_i) + \psi(t, t_i)u(t_i)$$

$$+ \int_{t_i}^{t} \Phi(t, \tau)G(\tau)dw(\tau) \tag{A-9}$$

where the matrix $\psi(t, t_i)$ is defined as

$$\psi(t, t_i) = \int_{t_i}^{t} \Phi(t, \tau)B(\tau)d\tau \tag{A-10}$$

Differentiating this with respect to time yields the equivalent defining relations

$$\frac{d}{dt}\psi(t, t_i) = B(t) + F(t)\psi(t, t_i) \tag{A-11}$$

$$\psi(t_i, t_i) = \int_{t_i}^{t_i} \Phi(t, \tau)B(\tau)d\tau = 0 \tag{A-12}$$

Thus, if an equation of the form

$$x_{i+1} = \Phi_{i+1/i}x_i + \psi_{i+1/i}u_i + \Gamma_{i+1/i}w_i \tag{A-13}$$

is to duplicate the state at the (i+1) sample time, $x(t_{i+1})$, as given by Equation (A-9), then it can be seen that

$$\Phi_{i+1/i} = \Phi(t_{i+1}, t_i) \tag{A-14}$$

$$\psi_{i+1/i} = \psi(t_{i+1}, t_i) \tag{A-15}$$

For filter applications, the last term in (A-13) need not be evaluated. Rather, an expression for the covariance of its contribution is required:

$$\Gamma_{i+1/i} Q_i \Gamma^T_{i+1/i} = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau) G(\tau) Q(\tau) G^T(\tau) \Phi^T(t_{i+1}, \tau) d\tau \tag{A-16}$$

or

$$\Gamma_{i+1/i} Q_i \Gamma^T_{i+1/i} = W(t_{i+1}, t_i) \tag{A-17}$$

where Equation (A-17) serves as a definition of $W(t_{i+1}, t_i)$. As in the previous case, a more convenient form for evaluation is obtained by differentiating Equation (A-16) to yield

$$\frac{d}{dt} W(t,t_i) = F(t)W(t,t_i) + W(t,t_i)F^T(t) + G(t)Q(t)G^T(t) \tag{A-18}$$

$$W(t_i,t_i) = 0 \tag{A-19}$$

Thus, to obtain the equivalent discrete model for a continuous-time system, Equations (A-7), (A-11), and (A-18) are integrated forward from the initial conditions (A-8), (A-12), and (A-19) to time t_{i+1}. The required matrices $\Phi_{i+1/i}$, $\psi_{i+1/i}$, and $\Gamma_{i+1/i}Q_i\Gamma^T_{i+1/i}$ are then determined from Equations (A-14), (A-15), and (A-17), respectively.

The influence of uncertain parameters in F(t) or B(t) upon the discrete model can be expressed analytically in simple cases. In more complex situations, the dependence can be found by numerical integration of these equations for various values of the parameters, from which functional relationships between the elements of $\Phi_{i+1/i}$, $\psi_{i+1/i}$, and $\Gamma_{i+1/i}Q_i\Gamma^T_{i+1/i}$ and the parameter values can be established. If the system is assumed to be time invariant and noise inputs are assumed to be stationary over N samples, a single set of integrations suffices for all sample periods.

For certain applications in which the system matrices are slowly varying or time invariant and in which the sample period is short compared to the system's natural transients, a first-order approximation to the solution of the differential equations may be adequate. These approximations are:

$$\Phi_{i+1/i} \cong I + F(t_i)(t_{i+1} - t_i) \qquad \text{(A-20)}$$

$$\psi_{i+1/i} \cong B(t_i)(t_{i+1} - t_i) \qquad \text{(A-21)}$$

$$\Gamma_{i+1/i} Q_i \Gamma^T_{i+1/i} \cong G(t_i) Q(t_i) G^T(t_i)(t_{i+1} - t_i) \qquad \text{(A-22)}$$

APPENDIX B

MINIMUM ERROR VARIANCE ESTIMATE

Consider a filter estimate x_i^* that is to be defined from measurements $z_1, z_2 \ldots z_i$ so that it minimizes the trace of the error variance of the estimate. Equationally, this is

$$tr\left(E\left[(x_i - x_i^*)(x_i - x_i^*)^T\right]\right) = \text{minimum} \tag{B-1}$$

or

$$E\left[(x_i - x_i^*)^T (x_i - x_i^*)\right] = \text{minimum} \tag{B-2}$$

To derive this estimate, the error variance must first be written in terms of the density function

$$E\left[(x_i - x_i^*)^T (x_i - x_i^*)\right] = \int\int \cdots \int (x_i - x_i^*)^T (x_i - x_i^*) \, p(x_i, z_1 \ldots z_i) \, dx_i dz_1 \ldots dz_i \tag{B-3}$$

where each integral sign represents a multiple integral since x_i, $z_1 \ldots z_i$ may be vectors. The density function can be written using Bayes' theorem as

$$p(x_i, z_1 \ldots z_i) = p(x_i | z_1 \ldots z_i) p(z_1 \ldots z_i) \tag{B-4}$$

Equation (B-3) is then

$$E\left[(x_i - x_i^*)^T (x_i - x_i^*)\right] = \int \cdots \int \left| (x_i - x_i^*)^T (x_i - x_i^*) p(x_i | z_1 \ldots z_i) \, dx_i \right|$$

$$p(z_1 \ldots z_i) \, dz_1 \ldots dz_i \tag{B-5}$$

Consider now the integral in brackets. Since x_i^* does not involve x_i, the integral in brackets is

$$\int (x_i - x_i^*)^T (x_i - x_i^*) p(x_i | z_1 \ldots z_i) \, dx_i =$$

$$E[x_i^T x_i | z_1 \ldots z_i] - E[x_i | z_1 \ldots z_i]^T x_i^*$$

$$- x_i^{*T} E[x_i | z_1 \ldots z_i] + x_i^{*T} x_i^*$$

$$= E[x_i^T x_i | z_1 \ldots z_i] - E[x_i | z_1 \ldots z_i]^T E[x_i | z_1 \ldots z_i]$$

$$+ \left| E[x_i | z_1 \ldots z_i] - x_i^* \right|^T \left| E[x_i | z_1 \ldots z_i] - x_i^* \right| \tag{B-6}$$

By definition, this quantity is positive. Therefore, to minimize the multiple integral, it is sufficient to choose x_i^* to minimize the integrand given by Equation (B-6). Only the last term involves x_i^* and the smallest value it can assume is zero. Thus, the minimizing estimate is given by

$$x_i^* = E[x_i | z_1 \ldots z_i] \tag{B-7}$$

The Kalman filter estimate $\hat{x}_i$ for linear Gaussian systems is also given by

$$\hat{x}_i^* = E[x_i | z_1 \ldots z_i] \tag{B-8}$$

Therefore, the Kalman filter estimate is a minimum error variance estimator (i.e., $\hat{x}_i = x_i^*$).

APPENDIX C

GAIN RELATIONSHIP FOR PERFECT MEASUREMENTS

If the measurements are directly related to the state in such a manner that ${}_{\alpha}H_i$ is zero, the modified filter gain is given by the expression

$$G_i = J_i(H_iJ_i + R_i)^{-1} \qquad \text{(C-1)}$$

When the noise covariance R_i is zero, the measurements determine the state exactly and the filter gain becomes

$$G_i = J_i(H_iJ_i)^{-1} \qquad \text{(C-2)}$$

or

$$H_iG_i = H_iJ_i(H_iJ_i)^{-1} \qquad \text{(C-3)}$$

$$= I$$

Premultiplying by H_i^T and performing the inverse operation gives

$$(H_i^T H_i) G_i = H_i^T \tag{C-4}$$

$$G_i = (H_i^T H_i)^{-1} H_i^T \tag{C-5}$$

Postmultiplying by H_i gives

$$G_i H_i = (H_i^T H_i)^{-1} H_i^T H_i = I \tag{C-6}$$

Some of the filter algorithms include the term $I - G_i H_i$. Thus, when the measurements are perfect, these terms will be zero.

APPENDIX D

DERIVED DENSITY THEOREM

Let z and ν be random N vectors where z equals $f(\nu)$ and ν equals $f^{-1}(z)$ with f and f^{-1} continuously differentiable. For an arbitrary set S

$$\Pr(z \epsilon S) = \Pr\left(f(\nu) \epsilon S\right)$$

$$= \Pr\left(\nu \epsilon f^{-1}(S)\right) \qquad \text{(D-1)}$$

where Pr( ) is the probability of ( ). In terms of the probability density function of z, $p_z(z)$, this probability can be written

$$\Pr(z \epsilon S) = \int_S p_z(z)\,dz \qquad \text{(D-2)}$$

In addition, the right half of Equation (D-1) can be written using the probability density function $f_\nu(\nu)$ as

$$\Pr[\nu \epsilon f^{-1}(S)] = \int_{f^{-1}(S)} p_\nu(\nu)\,d\nu$$

$$= \int_S p_\nu\left|f^{-1}(z)\right| \left\|\frac{\partial f^{-1}(z)}{\partial z}\right\| dz \qquad \text{(D-3)}$$

where the last integral follows from the rule of transformation of integrals with $\|\partial f^{-1}(z)/\partial z\|$ equal to the absolute value of the Jacobian determinant.

Combining these equations gives

$$\int_S \left[p_z(z) - p_\nu\left(f^{-1}(z)\right)\left\| \frac{\partial f^{-1}(z)}{\partial z}\right\|\right] dy = 0 \qquad \text{(D-4)}$$

Since this equation must hold for any arbitrary region of integration no matter how small or large, the integrand must be zero. Therefore, the density function of z derived from the density function of ν is

$$p_z(z) = p_\nu\left|f^{-1}(z)\right|\left\|\frac{\partial f^{-1}(z)}{\partial z}\right\| \qquad \text{(D-5)}$$

APPENDIX E

EVALUATION OF FISHER INFORMATION MATRIX

The Fisher information matrix developed in Chapter 4 is given by the expression

$$F(\alpha_k,\alpha_\ell) = \sum_i^N \Bigg\{ 4E\left[\frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \frac{\partial \nu_i}{\partial \alpha_\ell}\right]$$

$$+ E\left[\nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i\right]$$

$$- \operatorname{tr}(V^{-1} \frac{\partial V_i}{\partial \alpha_k}) \operatorname{tr}(V^{-1} \frac{\partial V_i}{\partial \alpha_\ell}) \Bigg\} \qquad \text{(E-1)}$$

Determination of $F(\alpha_k,\alpha_\ell)$ requires an evaluation of the second term of this expression. This evaluation is accomplished by letting the innovation ν_i be defined in terms of a vector Λ whose components are independent Gaussian variables with zero mean and unity variance. The desired relationship is

$$\nu_i = \sqrt{V_i}\, \Lambda \qquad \text{(E-2)}$$

so that

$$E(\nu_i \nu_i^T) = \sqrt{V_i}\ E[\Lambda\Lambda^T]\ \sqrt{V_i^T}$$

$$= \sqrt{V_i}\ I \sqrt{V_i}$$

$$= V_i \tag{E-3}$$

which is consistent with the defined variance of ν_i. Also, let symmetric matrices ${}_{\alpha_k}S'$ and ${}_{\alpha_\ell}S''$ be defined by

$${}_{\alpha_k}S' = \sqrt{V_i}^T\ V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1}\ \sqrt{V_i} \tag{E-4}$$

$${}_{\alpha_\ell}S'' = \sqrt{V_i}^T\ V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1}\ \sqrt{V_i} \tag{E-5}$$

The second term of the information matrix becomes

$$E\left[\nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i\right] = E\left[\Lambda^T {}_{\alpha_k}S'\Lambda\Lambda^T {}_{\alpha_\ell}S''\Lambda\right]$$

$$= E\left[tr({}_{\alpha_k}S'\Lambda\Lambda^T)\, tr({}_{\alpha_\ell}S''\Lambda\Lambda^T)\right] \tag{E-6}$$

Since the components of Λ, Λ_i, are independent, first and third moments are zero and second and fourth moments are 1 and 3, respectively. Therefore, the expected value above can be written

$$E\left[\mathrm{tr}({}_{\alpha_k}S'\Lambda\Lambda^T)\,\mathrm{tr}({}_{\alpha_\ell}S'\Lambda\Lambda^T)\right] = \mathrm{tr}({}_{\alpha_k}S')\,\mathrm{tr}({}_{\alpha_\ell}S'') + 2\,\mathrm{tr}({}_{\alpha_k}S'\,{}_{\alpha_\ell}S'') \tag{E-7}$$

The second term of the information matrix then becomes

$$E\left[\nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \nu_i \nu_i^T V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \nu_i\right] =$$

$$\mathrm{tr}\left(\sqrt{V_i^T}\, V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \sqrt{V_i}\right) \mathrm{tr}\left(\sqrt{V_i^T}\, V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \sqrt{V_i}\right)$$

$$+\ 2\,\mathrm{tr}\left(\sqrt{V_i^T}\, V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \sqrt{V_i}\,\sqrt{V_i^T}\, V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell} V_i^{-1} \sqrt{V_i}\right)$$

$$= \mathrm{tr}\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right) \mathrm{tr}\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right) + 2\,\mathrm{tr}\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \frac{\partial V_i}{\partial \alpha_\ell}\right) \tag{E-8}$$

Substituting into Equation (E-1) and utilizing the sample data yields

$$F(\alpha_k,\alpha_{\ell}) = \sum_i^N \left\{ 4E\left[\frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \frac{\partial \nu_i}{\partial \alpha_{\ell}}\right] + tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right) tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_{\ell}}\right)\right.$$

$$\left. + 2tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \frac{\partial V_i}{\partial \alpha_{\ell}}\right) - tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k}\right) tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_{\ell}}\right)\right\}$$

$$= \sum_i^N \left\{ 4 \frac{\partial \nu_i^T}{\partial \alpha_k} V_i^{-1} \frac{\partial \nu_i}{\partial \alpha_{\ell}} + 2tr\left(V_i^{-1} \frac{\partial V_i}{\partial \alpha_k} V_i^{-1} \frac{\partial V_i}{\partial \alpha_{\ell}}\right)\right\} \tag{E-9}$$

which completes the evaluation of the Fisher information matrix.

APPENDIX F

PARAMETRIC LIMITS OF SPINNING BODIES

The basic parameters of spinning bodies are: the angular momentum of the body M; the kinetic energy of the body T; and the moments of inertia for the three principal axes of the satellite I_1, I_2, and I_3. The moments of inertia define body shape constraints while the angular momentum, kinetic energy and moments of inertia dictate limits of rigid body motion.

F.1 MOMENT OF INERTIA CONSTRAINTS

The moments of inertia about each axis are the integrated products over the body of the mass of each element in the body and the square of the distance from the axis to the mass element. The minimum moment of inertia I_1 is about the $\hat{\underline{b}}_1$ axis, the intermediate value I_2 is about the $\hat{\underline{b}}_2$ axis, and the maximum value I_3 is about the $\hat{\underline{b}}_3$ axis (see Figure F-1). If r_1 is the generic scalar component along the $\hat{\underline{b}}_1$ axis to a mass element dm, and r_2 and r_3 are generic scalar components along the $\hat{\underline{b}}_2$ and $\hat{\underline{b}}_3$ axes so that the generic vector is $\underline{r} = r_1\hat{\underline{b}}_1 + r_2\hat{\underline{b}}_2 + r_3\hat{\underline{b}}_3$, then the moments of inertia are

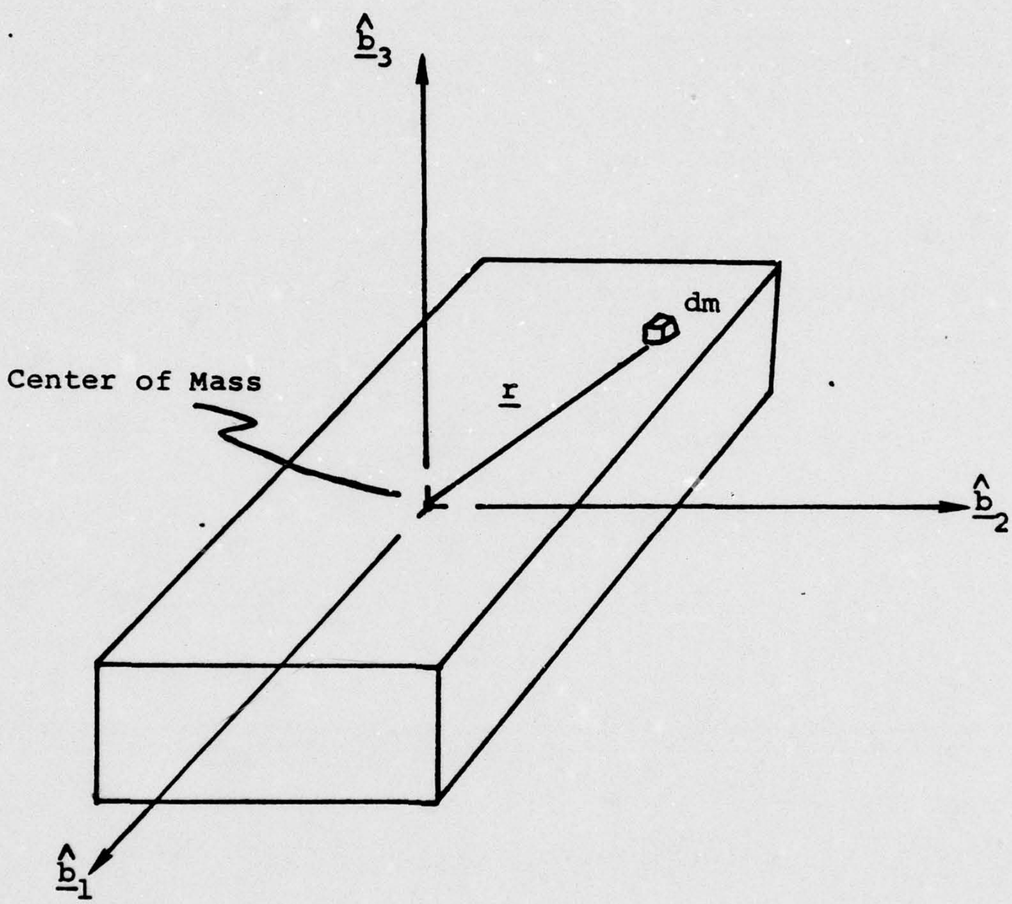

Figure F-1. Body Axes.

$$I_1 = \int (r_2^2 + r_3^3)\ dm \tag{F-1}$$

$$I_2 = \int (r_1^2 + r_3^2)\ dm \tag{F-2}$$

$$I_3 = \int (r_1^2 + r_2^2)\ dm \tag{F-3}$$

Adding Equations (F-1) and (F-2) gives

$$I_1 + I_2 = \int (r_1^2 + r_2^2)\ dm + 2\int r_3^2\ dm \tag{F-4}$$

or

$$I_1 + I_2 = I_3 + 2\int r_3^2\ dm \tag{F-5}$$

Equation (F-5) leads to the inequality

$$I_1 + I_2 \geq I_3 \tag{F-6}$$

or

$$\frac{I_1}{I_3} + \frac{I_2}{I_3} \geq 1 \tag{F-7}$$

This condition is illustrated by the lined area in Figure F-2. The equality of Equation (F-7) represents the limiting body shape, which is a flat plate with negligible thickness.

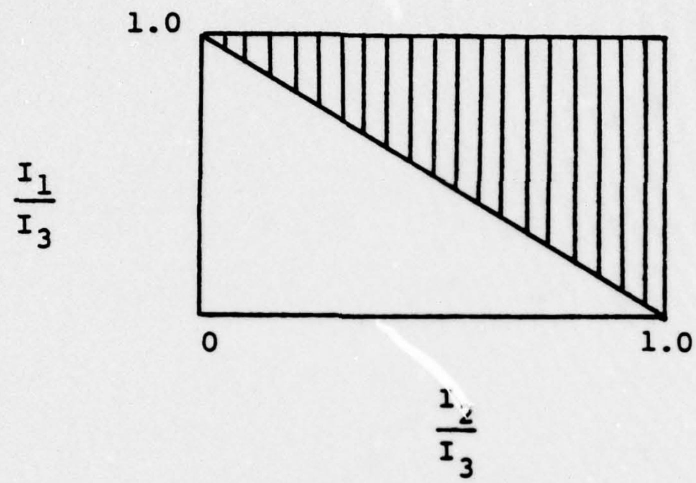

Figure F-2. Constraint on Moments of Inertia ($I_1+I_2 \geq I_3$).

The moments of inertia are defined as

$$I_3 \geq I_2 \geq I_1 \quad . \tag{F-8}$$

or

$$1 \geq \frac{I_2}{I_3} \geq \frac{I_1}{I_3} \tag{F-9}$$

This condition is illustrated by the lined area in Figure F-3.

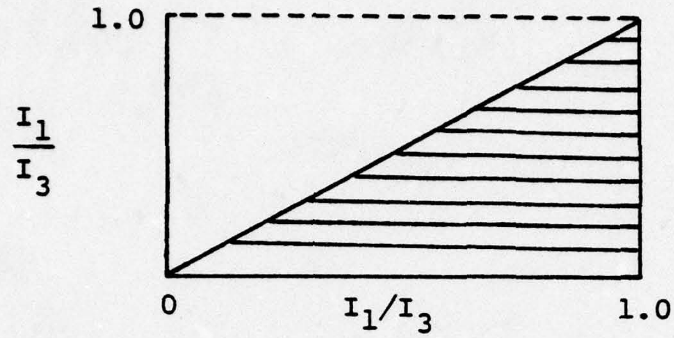

Figure F-3. $(I_3 \geq I_2 \geq I_1)$ Moment of Inertia Constraint.

Superimposing Figure F-3 on Figure F-2 gives Figure F-4 which shows the two constraints on the ratios I_2/I_3 and I_1/I_3. The double-lined area in Figure F-4 reveals the boundary limits.

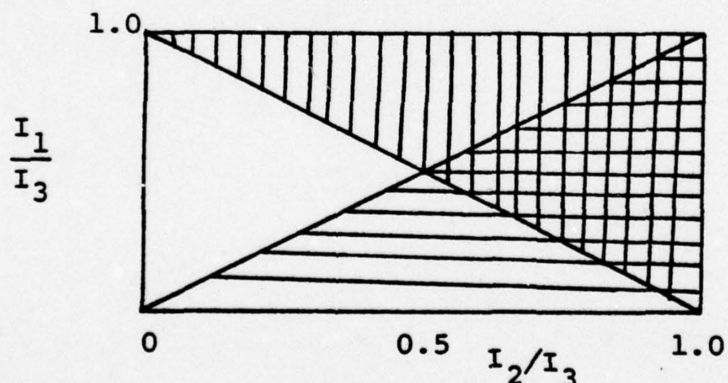

Figure F-4. Total Constraints on Moments of Inertia.

The ratio I_2/I_3 is bounded by the values 0.5 and 1.0, while the bound for I_1/I_3 depend on the value of I_2/I_3. The upper extreme is I_2/I_3 or a symmetrically shaped body

and the lower limit is $1 - I_2/I_3$ or a flat plate with negligible thickness.

F.2 RIGID BODY MOTION CONSTRAINTS

The general motion of a spinning rigid body of symmetry in the absence of applied moments can be described by steady precession or coning of the symmetry axis (or spin axis) about the angular momentum vector M, which is fixed in inertial space (see Figure F-5). The term spin axis is applied to the body axis with which the angular velocity vector is nominally aligned for steady spinning motion, with no precession. For an axisymmetric body, the symmetry axis must be selected as the spin axis.

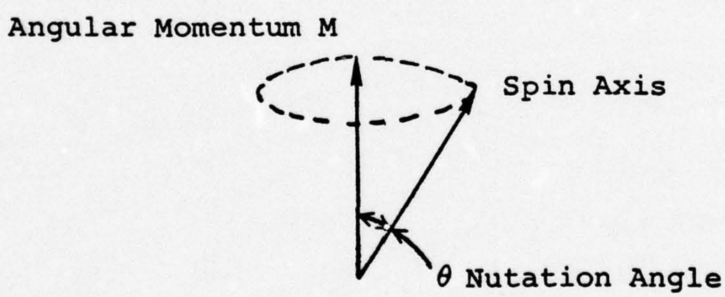

Figure F-5. Vector Representation for Free Body Motion.

The angle θ between the spin axis and the angular momentum vector is called the nutation angle or coning angle and remains constant for a rigid body of symmetry during the precession cycle. An asymmetric body also

exhibits precession of the spin axis but, in addition, displays an oscillation in θ during the precession cycle that is called nutation. The frequency of oscillation in nutation is twice that of precession.

(The above definitions for precession and nutation are based on etymology and classical usage. In modern applications of space sciences and gyrodynamics other usages are common. Unfortunately, with regard to axisymmetric bodies the word nutation is often applied to the coning motion characterized here as precession.)

With the origin of body axes $\hat{\underline{b}}_1$, $\hat{\underline{b}}_2$, and $\hat{\underline{b}}_3$ coinciding with the body center of mass, the rotational equations of motion for a rigid body in the absence of applied torques are

$$I_1\dot{\omega}_1 - (I_2-I_3)\ \omega_2\omega_3 = 0 \tag{F-10}$$

$$I_2\dot{\omega}_2 - (I_3-I_1)\ \omega_1\omega_3 = 0 \tag{F-11}$$

$$I_3\dot{\omega}_3 - (I_1-I_2)\ \omega_1\omega_2 = 0 \tag{F-12}$$

where ω_1, ω_2, and ω_3 are the angular velocities about the respective body axes and I_1, I_2, and I_3 are the principal moments of inertia defined so that $I_3 \geq I_2 \geq I_1$. Multiplying Equations (F-10) through (F-12) by ω_1, ω_2, and ω_3, respectively, and adding and then integrating yields

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T \qquad \text{(F-13)}$$

where the constant of integration 2T is twice the kinetic energy of rotation. In an analogous way, multiplying Equations (F-10) through (F-12) by $I_1\omega_1$, $I_2\omega_2$, and $I_3\omega_3$, adding and integrating gives

$$(I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2 = M^2 \qquad \text{(F-14)}$$

where M is the angular momentum.

With the equations, the motion of a rigid body spinning about the axis of maximum inertia can be stated by the conditional equation

$$2T\, I_3 \geq M^2 \geq 2T\, I_2 \qquad \text{(F-15)}$$

or

$$1 \geq \frac{M^2}{2T\ I_3} > \frac{I_2}{I_3} \tag{F-16}$$

The geometric interpretation of Equation (F-16) can be illustrated by considering the nutation angle. The upper limit of $M^2/2T\ I_3$ implies that the nutation angle is zero (i.e., pure spin about the maximum inertia axis) while the lower limit implies that the nutation angle approaches a maximum value of 90 deg. Figure F-6 illustrates the variation in the limit of the nutation angle for values of I_2/I_3 and $M^2/2T\ I_3$. As the value of $M^2/2T\ I_3$ decreases from one, the nutation angle increases from 0 to 90 deg.

For a symmetric body (i.e., $I_1 = I_2$) with constant spin about the axis of maximum moment of inertia ω_3, Equations (F-10) to (F-12) can be used to show that the spin axis moves with respect to the body axes about the angular momentum vector at a constant nutation angle with a frequency Ω given by

$$\Omega = \frac{(I_3 - I_1)\ \omega_3}{I_1} \tag{F-17}$$

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In addition, since the moments of inertia are defined so that I_3 is greater than I_1, the spin axis will rotate about the fixed angular momentum vector in retrograde precession.

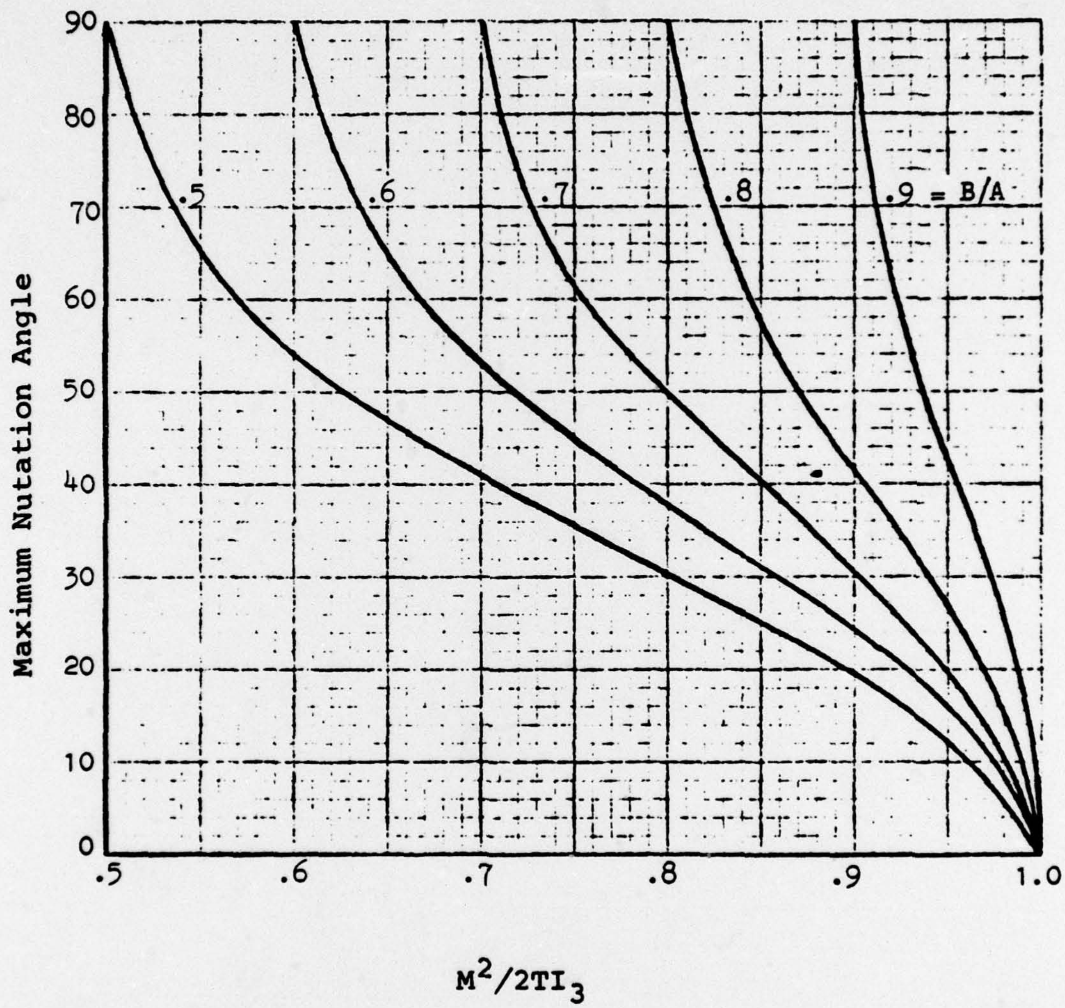

Figure F-6. Maximum Nutation Angle versus $M^2/2TI_3$.

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